Collusion-proof Mechanisms for Full Surplus Extraction

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Abstract

The paper examines information structures that can guarantee full surplus extraction via collusion-proof mechanisms. Our collusion-proofness notion requires that there does not exist any coalition whose manipulation can affect the mechanism designer's payoff. When the mechanism designer is restricted to using standard Bayesian mechanisms, we show that under almost every prior distribution of agents' types, there exist payoff structures under which there is no collusion-proof full surplus extracting mechanism. However, when ambiguous mechanisms are allowed, we provide a weak necessary and sufficient condition on the prior such that collusion-proof full surplus extraction can be guaranteed. Thus, the paper sheds light on how the collusion-proofness requirement resolves the full surplus extraction paradox of Crémer and McLean (1985, 1988) and how engineering ambiguity in mechanism rules restores the paradox.

Keywords: Collusion-proofness; Multiple coalitions; Full surplus extraction; Bayesian mechanism; Ambiguous mechanism; Correlated beliefs.

JEL: D81; D82.

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1 Introduction

In mechanism design theory, most works assume that agents (he) behave noncooperatively when revealing their private information to the mechanism designer (MD, she). However, there are many real-life mechanisms, such as auctions and voting, where collusion arises as a common practice, and the MD has limited power in banning it or detecting its makeup.[1] When a group of agents sees room to profit from collusion, imposing individual incentive constraints on the mechanism alone may not ensure the MD's desired outcome. In response, we wish to design mechanisms that are immune from all coalitions' joint manipulations.

This paper explores what information structures can guarantee full surplus extraction (FSE) via collusion-proof mechanisms. FSE mechanisms, giving the MD the first-best total surplus and leaving agents zero rent, are rarely seen in practice. Crémer and McLean (1985, 1988) have characterized information structures that guarantee FSE: such information structures necessarily involve correlated private information (types), yet exist broadly in any finite-dimensional type space. As a result, the theoretically permissive result on FSE is often interpreted as a paradox. In Crémer and McLean (1985, 1988)'s FSE mechanisms, each agent's monetary transfer is highly sensitive to other agents' types reported to the MD, which makes it natural for agents to contemplate colluding. Hence, it is of interest to study the extent to which the collusion-proofness requirement restricts the information structures that guarantee FSE and serves as a resolution of the paradox.

Our collusion-proofness notion is related to the ones proposed by Laffont and Martimort (2000) and Che and Kim (2006). Similar to these papers, we assume that members of a coalition face information asymmetry and can only collude via an incentive compatible and individually rational side contract. We consider a general model with at least two players and assume that all coalitions can be formed, as in notions of the strong Nash equilibrium and the core. This differentiates our model from the two-agent two-type case studied by Laffont and Martimort (2000), and contrasts with the scenario where the MD knows a specific coalition that she should be concerned about as in Che and Kim (2006). Specifically, our robust collusion-proofness (RCP) condition mandates that the collusion of any coalition, if occurs,

[1] See Che and Kim (2006) for a review of the literature on collusion against exogenously given institutions.

should not harm the MD, which is the same as the one of Che and Kim (2006), except that we require the mechanism to be immune from manipulations of all coalitions.

Although FSE can be guaranteed under a broad set of information structures, we find that achieving it via collusion-proof standard Bayesian mechanisms is challenging. In particular, Proposition 1 shows that under some mild dimensional restrictions on agents' type space, for almost all prior distributions over it, there exist payoff structures under which it is impossible to achieve FSE via a mechanism satisfying the RCP condition. The findings contrast with the result in Che and Kim (2006): in a model with at least three agents where the grand coalition can collude, they show that there is a broad class of information structures for which collusion-proof FSE can be guaranteed. Thus, the peril that more coalitions may be formed significantly reduces information structures that guarantee FSE and offers a resolution of the FSE paradox.

As it is difficult to guarantee collusion-proof FSE by adopting standard Bayesian mechanisms, the MD might be motivated to use a broader collection of tools called ambiguous mechanisms (e.g., Bose and Renou, 2014; Di Tillio et al., 2017; Guo, 2019). An ambiguous mechanism has vaguely described rules: the MD can secretly commit to a standard Bayesian mechanism, but strategically announce multiple potential mechanisms. In reality, the vague tax audit scheme can be viewed as an ambiguous mechanism. We assume that agents are ambiguity-averse towards the unknown mechanism rule and make decisions with the maxmin expected utility of Gilboa and Schmeidler (1989).

We show that collusion-proof FSE via ambiguous mechanisms can be guaranteed, if agents' prior distribution satisfies the Coalition Beliefs Determine Preferences (CBDP) property. This is true under the collusion-proofness notion of RCP generalized to ambiguous mechanisms. Moreover, we slightly strengthen the RCP condition into the RCP* condition, where the latter allows members of a coalition to utilize a side contract that is also contingent on information revealed from the main mechanism. Then collusion-proof FSE in the sense of RCP* can be guaranteed via ambiguous mechanisms, if and only if the prior satisfies the CBDP property. The CBDP property strengthens the Beliefs Determine Preferences (BDP) property of Neeman (2004) by also requiring the knowledge of any non-grand coalition's posterior belief over types of agents out of the coalition to pin down this coalition's type profile. In any fixed finite type space, the CBDP property imposes a weak restriction on the prior over the type space. Thus, there is a broad class of prior distributions under which collusion-proof FSE can be achieved via ambiguous mechanisms. In particular, ambiguity can be engineered to soften the unpermissive result on collusion-proof FSE in the two-agent environment of Laffont and Martimort (2000), and to address the peril that all coalitions can be formed in the multiple-agent model of Che and Kim (2006). Therefore, the use of ambiguous mechanisms can restore the FSE paradox.

Literature Review. The paper is related to three strands of the literature.

First, the paper is related to the literature on mechanism design under correlated beliefs.

Among others, Crémer and McLean (1985, 1988), McAfee and Reny (1992), and Lopomo et al. (2022) have characterized conditions on the information structures so that FSE can be guaranteed. In particular, in a finite type space, Crémer and McLean (1988) have shown that Convex Independence is the necessary and sufficient condition on the prior to guarantee FSE. A related question is what information structures with correlated beliefs can guarantee the implementability of all efficient allocations, with or without additional individual rationality and/or budget balance restrictions on the mechanism. The proper scoring rule (see Börgers et al., 2015, for a reference) and the work of d'Aspremont et al. (1990, 2004), McLean and Postlewaite (2004, 2015), and Kosenok and Severinov (2008), among others, have provided answers to these questions. The methodology adopted in the current paper is related to Crémer and McLean (1988) and Kosenok and Severinov (2008): we also focus on a finite type space and establish the existence of a desirable mechanism via the duality approach. A key difference is that we design the mechanism in a way that is immune from collusion.

The current paper is directly related to the literature on collusion-proof mechanisms.

One approach in the literature considers all possible coalitions' deviations and imposes the collusion-proofness requirement on the mechanism axiomatically without explicitly modeling strategic interactions due to information asymmetry within a coalition. Green and Laffont (1979), Chen and Micali (2012), Bierbrauer and Hellwig (2016), and Safronov (2018), among others, adopt this approach. This approach provides a benchmark to study collusion-proof mechanisms since the worst-case scenario from the MD's perspective is often that all coalitions may be formed and that agents collude without encountering information frictions.

In this strand of the literature, Green and Laffont (1979), Chen and Micali (2012), and Bierbrauer and Hellwig (2016) focus on ex-post collusion-proofness notions, and Safronov (2018) adopts an interim notion. In particular, Safronov (2018) shows that in private-value environments with independent beliefs, every efficient allocation rule is implementable via an incentive compatible, budget balanced, and collusion-proof mechanism. We visit the collusion-proofness notion of Safronov (2018) in the Online Appendix as a supplement of the RCP condition considered in the paper. The message that it is difficult to guarantee collusion-proof FSE under standard Bayesian mechanisms, but easy under ambiguous mechanisms, remains true.

Another approach to studying collusion-proof mechanisms focuses on one particular coalition that can be formed and explicitly considers within-coalition information asymmetry, which may undermine the coalition's ability to collude. Laffont and Martimort (1997, 2000), Che and Kim (2006), and Meng et al. (2017) follow this approach. Laffont and Martimort (1997, 2000) and Meng et al. (2017) characterize the optimal collusion-proof mechanisms in their specific payoff structures with two agents and two types. One observation from them is that the collusion-proofness requirement does not have an additional bite into the MD's ability to extract agents' surplus in the independent belief case, but can have an additional bite if agents have correlated beliefs. Che and Kim (2006) investigate environments with more agents and general payoff structures. They extend the positive result on collusion-proof mechanism design under independent beliefs and also identify a sufficient condition on the information structure that guarantees collusion-proof FSE. Our RCP and RCP* conditions are built on the collusion-proofness notion of Che and Kim (2006): we strengthen their notion by requiring a mechanism to be immune from all coalitions' manipulations. Our work contributes to this approach by revealing the importance of the MD's knowledge of the composition of the colluding coalition in the FSE problem: when all coalitions can be formed, it may be impossible to design a standard Bayesian mechanism that achieves FSE and satisfies the RCP condition.

The paper also fits into the literature on mechanism design with ambiguity-averse agents.

Some works in this literature, including the current one, explore if it is possible to strategically engineer ambiguity in the mechanism to improve its performance. One approach involves endogenously generating ambiguity in agents' beliefs towards other agents' types: Bose and Renou (2014) do so via an ambiguous communication device and the endogenously engineered ambiguous beliefs may allow the MD to implement social choice functions that are not implementable otherwise. A second approach generates ambiguity on the payoff rule more directly. For example, Di Tillio et al. (2017), Bose and Daripa (2017), Guo (2019), and Tang and Zhang (2021) demonstrate that ambiguous mechanisms are more potent than standard Bayesian mechanisms in screening, preferences elicitation, FSE, and implementing social choice correspondences. To the best of our knowledge, the current paper is the first one that studies how ambiguous mechanisms can be introduced to address collusion.

In some other works, agents are assumed to hold ambiguous beliefs about other agents' types exogenously, and the MD designs the optimal/efficient standard Bayesian mechanisms, e.g., Bose et al. (2006), Bose and Daripa (2009), Renou (2015), Wolitzky (2016), De Castro and Yannelis (2018), Song (2018, 2023), Kocherlakota and Song (2019), and Lopomo et al. (2020). The current work differs from these papers, as we do not assume ambiguous beliefs about other agents' types.

The rest of the paper proceeds as follows. Section 2 sets up the model. Section 3 provides a motivating example. Section 4 defines the collusion-proofness notion and provides a theoretically unpermissive resuls on collusion-proof FSE via standard Bayesian mechanism. Section 5 presents a possibility result by adopting ambiguous mechanisms. Section 6 concludes. All proofs are relegated to the Appendix.

2 Set-up

We study an environment with one mechanism designer (MD, she) and a finite set of agents (he) $I = \{1, 2, ..., n \geqslant 2\}$. Each $i \in I$ privately observes his type $\theta_i \in \Theta_i$, where Θ_i is i's type set satisfying $2 \leqslant |\Theta_i| < +\infty$ and $\Theta \equiv \prod_{i \in I} \Theta_i$ is the finite type space. Assume that there is a full-support common prior over Θ, i.e., a $p \in \Delta(\Theta)$ such that $p(\theta) > 0$ for each $\theta \in \Theta$. The pair (Θ, p) is called an **information structure**.

For any $S \in 2^I \backslash \{\varnothing\}$, S is an agent if $|S| = 1$, a **coalition** if $2 \leqslant |S| \leqslant n$, and the grand coalition if $|S| = n$. For S with $1 \leqslant |S| < n$, given type profile $\theta_S \equiv (\theta_i)_{i \in S} \in \Theta_S \equiv \prod_{i \in S} \Theta_i$, we let $p(\cdot|\theta_S) \equiv (p(\theta_{-S}|\theta_S))_{\theta_{-S}\in\Theta_{-S}}$ denote the posterior belief over types of agents out of S, where $p(\theta_{-S}|\theta_S) \equiv \frac{p(\theta_S,\theta_{-S})}{p(\theta_S)}$, $p(\theta_S) \equiv \sum_{\theta'_{-S}\in\Theta_{-S}} p(\theta_S,\theta'_{-S})$, and $\theta_{-S} \equiv (\theta_j)_{j\in I\setminus S}$. Agents are said to have independent beliefs, if for all $i \in I$, $p(\cdot|\theta_i)$ is constant across different $\theta_i \in \Theta_i$; otherwise, agents have correlated beliefs. For simplicity, denote θ_I by θ.

The MD's quasi-linear utility function is of the form $u_0(a) - \sum_{i\in I} \bar{t}_i$ and each agent's quasi-linear utility function is of the form $u_i(a,\theta) + \bar{t}_i$. We let a denote a feasible outcome and A be a compact set of feasible outcomes that contains all lotteries over feasible pure outcomes. Let $u_0(a)$ and $u_i(a,\theta)$ be the MD's and agent i's payoff from outcome a, respectively, and $\bar{t}_i \in \mathbb{R}$ be the monetary transfer from the MD to agent i.[2] We call the profile of utility functions $(u_0, (u_i)_{i\in I})$ a **payoff structure**.

For example, in the single-unit auction case, we may have $A = \Delta\{0, 1, ..., n\}$, where these pure outcomes mean that the good is not produced, allocated to agent 1,..., and allocated to agent n, respectively. View $-u_0(a)$ as the MD's cost of producing outcome a. For each $i \in I$, $u_i(a,\theta)$ may depend on all agents' private information (e.g., as in a common value auction), or in the degenerate private-value case, depend on θ_i only (e.g., in a private-value auction).

Let (q,t) denote a (direct) **standard Bayesian mechanism**, where $q : \Theta \to A$ is the allocation rule that assigns the outcome and $t : \Theta \to \mathbb{R}^n$ is the transfer rule that describes the monetary payment received by agents. We may simply call (q,t) a mechanism.

Given agent i's **reporting strategy** $\sigma_i : \Theta_i \to \Delta(\Theta_i)$, $\sigma_i[\theta_i](\theta'_i)$ is the probability that type-θ_i agent reports θ'_i. Let $\bar{\sigma}_i$ denote the truthful reporting strategy, i.e., the one such that $\bar{\sigma}_i[\theta_i](\theta_i) = 1$ for all $\theta_i \in \Theta_i$. If type-θ_i agent i follows strategy σ_i and other agents truthfully report, his utility is

$$V_i[q,t](\theta_i,\sigma_i) \equiv \sum_{\theta'_i\in\Theta_i} \sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\theta'_i,\theta_{-i}),(\theta_i,\theta_{-i})\big) + t_i(\theta'_i,\theta_{-i})]p(\theta_{-i}|\theta_i)\sigma_i[\theta_i](\theta'_i).$$

When $\sigma_i[\theta_i](\theta'_i) = 1$, we may let $V_i[q,t](\theta_i,\theta'_i)$ denote $V_i[q,t](\theta_i,\sigma_i)$ for simplicity.

We also introduce a parallel notation for coalition S for later convenience. For any

[2]We assume that $u_0(\cdot)$ and $u_i(\cdot,\theta)$ defined over lotteries are consistent with the expected utility theory. We also remark that the analysis of the paper does not change if we generalize u_0 so that it depends on both a and θ. However, we focus on the current setup, which is the same as that of Che and Kim (2006), to highlight our observation that their possibility result on collusion-proof FSE is overturned when all coalitions can be formed.

coalition S, we let $\delta^S : \Theta_S \to \Delta(\Theta_S)$ denote the **joint reporting strategy** of members in coalition S, under which $\delta^S[\theta_S](\theta'_S)$ is the probability that members with type profile θ_S jointly report θ'_S. For each coalition S, joint reporting strategy δ^S, and agent $i \in S$, we define

$$V_i[q,t](\theta_S, \delta^S) \equiv \sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} [u_i\big(q(\theta'_S, \theta_{-S}), (\theta_S, \theta_{-S})\big) + t_i(\theta'_S, \theta_{-S})]p(\theta_{-S}|\theta_S)\delta^S[\theta_S](\theta'_S)$$

and

$$V_S[q,t](\theta_S, \delta^S) \equiv \sum_{i \in S} V_i[q,t](\theta_S, \delta^S),$$

which are, respectively, the utility of agent i and the total utility of members in coalition S when S has type profile θ_S and follows joint reporting strategy δ^S. Also, let $V_i[q,t](\theta_S, \theta'_S)$ and $V_S[q,t](\theta_S, \theta'_S)$ denote the above two terms when $\delta^S[\theta_S](\theta'_S) = 1$. We say δ^S is deterministic if for each $\theta_S \in \Theta_S$, there exists θ'_S such that $\delta^S[\theta_S](\theta'_S) = 1$. When $|S| = 1$, i.e., S consists of an agent, we abuse notation and view δ^S as this agent's reporting strategy.

The mechanism (q,t) is said to be **feasible** if it satisfies the interim individual rationality (IR) condition and the interim incentive compatibility (IC) condition below:

IR $\quad V_i[q,t](\theta_i, \bar{\sigma}_i) \geqslant 0, \forall i \in I, \theta_i \in \Theta_i,$

IC $\quad V_i[q,t](\theta_i, \bar{\sigma}_i) \geqslant V_i[q,t](\theta_i, \sigma_i), \forall i \in I, \theta_i \in \Theta_i, \sigma_i : \Theta_i \to \Delta(\Theta_i).$

An allocation rule q is said to be **efficient**, if it maximizes the ex-ante total surplus, i.e.,

$$\sum_{\theta \in \Theta} [u_0\big(q(\theta)\big) + \sum_{i \in I} u_i\big(q(\theta), \theta\big)]p(\theta) = \max_{\tilde{q}:\Theta \to A} \sum_{\theta \in \Theta} [u_0\big(\tilde{q}(\theta)\big) + \sum_{i \in I} u_i\big(\tilde{q}(\theta), \theta\big)]p(\theta) \equiv FS,$$

where FS stands for the **full surplus**, or equivalently, if q maximizes the ex-post total surplus pointwise, i.e., $u_0\big(q(\theta)\big) + \sum_{i \in I} u_i\big(q(\theta), \theta\big) \geqslant u_0(a) + \sum_{i \in I} u_i\big(a, \theta\big)$ for all $a \in A$ and $\theta \in \Theta$.

Given a payoff structure $(u_0, (u_i)_{i \in I})$, we say (q,t) achieves **full surplus extraction** (FSE), if (q,t) is feasible and makes the MD's ex-ante payoff equal to the FS, i.e.,

$$\sum_{\theta \in \Theta} [u_0\big(q(\theta)\big) - \sum_{i \in I} t_i(\theta)]p(\theta) = FS.$$

FSE requires that q must be efficient and agents' IR constraints must bind. We say an information structure (Θ, p) **guarantees FSE** if for any payoff structure $(u_0, (u_i)_{i \in I})$, there exists a mechanism (q,t) that achieves FSE. Crémer and McLean (1988) have shown that an information structure (Θ, p) guarantees FSE, if and only if the prior p satisfies the Convex Independence condition (Definition 2 in Appendix A.1), which necessitates correlated beliefs. For almost all priors p over a finite type space Θ, Convex Independence is satisfied.

3 Revisiting Laffont and Martimort (2000)'s Example

In this section, we revisit a two-agent public good example of Laffont and Martimort (2000). In this example, there exists a standard Bayesian mechanism under which the MD extracts the full surplus from agents, but we reiterate the concern that the mechanism is susceptible to collusion. Then, we engineer ambiguity in the mechanism, and assume that agents are ambiguity averse with respect to the engineered ambiguity. With such a preference, the MD can use an ambiguous mechanism to extract the full surplus in a way immune from collusion.

Suppose the MD can provide $\bar{q}$ units of public good at the cost of $0.5\bar{q}^2$.[3] There are two agents $I = \{1, 2\}$. Each agent i has a utility $\theta_i \bar{q}$ from $\bar{q}$ units of public good, where $\theta_i \in \Theta_i \equiv \{\theta_i^H = 1, \theta_i^L = 0\}$. The ex-post total surplus of producing $\bar{q}$ units of public good is $(\theta_1 + \theta_2)\bar{q} - 0.5\bar{q}^2$, which implies that the efficient allocation rule $q : \Theta \to \mathbb{R}_+$ satisfies $q(\theta) = \theta_1 + \theta_2$ for all $\theta \in \Theta$. With the common prior over Θ in Table 1, the ex-ante total surplus is 0.8.

p	θ_2^H	θ_2^L
θ_1^H	0.3	0.2
θ_1^L	0.2	0.3

Table 1: Common prior p

(t_1, t_2)	θ_2^H	θ_2^L
θ_1^H	$(0, 0)$	$(-4, -6)$
θ_1^L	$(-6, -4)$	$(4, 4)$

Table 2: A transfer rule t

Notice that the information structure (Θ, p) satisfies the Convex Independence condition of Crémer and McLean (1988) (reviewed in Section A.1). Therefore, there exists a standard Bayesian mechanism that achieves FSE, e.g., the mechanism (q, t), where t is given by Table 2. We omit the process of verifying that (q, t) achieves FSE.

This mechanism is susceptible to group manipulation. When two agents have type profiles

[3] Assume that $\bar{q}$ is bounded above by some large number and bounded below by zero.

(θ_1^H, θ_2^H), (θ_1^H, θ_2^L), (θ_1^L, θ_2^H), and (θ_1^L, θ_2^L), their ex-post payoff profiles are $(2, 2)$, $(-3, -6)$, $(-6, -3)$, and $(4, 4)$, and the MD's ex-post payoffs are -2, 9.5, 9.5, and -8. Suppose the two agents collude by always reporting the type profile (θ_1^L, θ_2^L) to the MD instead — they have the incentive to do so as each of them will earn a constant ex-post payoff of 4 instead. However, the MD can no longer achieves FSE if this collusion occurs.

Of course, there are other standard Bayesian mechanisms that achieves FSE. Are all of them susceptible to group manipulation? Laffont and Martimort (2000) formalize a rather weak collusion-proofness notion. They show that in the current setting, there is no standard Bayesian mechanism that simultaneously achieves FSE and satisfies their collusion-proofness requirement.[4]

We define two transfer rules η and ϕ in Table 3 and use them to construct transfer rules $\hat{t} = \eta + \lambda\phi$ and $\hat{\hat{t}} = \eta - \lambda\phi$, where $\lambda \geqslant 0.8$ is a constant multiplier. Notice that the η part enters transfer rules $\hat{t}$ and $\hat{\hat{t}}$ in the same way, but the ϕ part is multiplied by λ or $-\lambda$. Also, the ϕ part is **ex-post budget balanced**, i.e., $\sum_{i \in I} \phi_i(\theta) = 0$ for all $\theta \in \Theta$, and thus does not affect the MD's ex-post payoff.

Now, we allow the MD to engineer ambiguity by announcing that she has committed to one standard Bayesian mechanism in $\{(q, \hat{t}), (q, \hat{\hat{t}})\}$ without further details on which one is chosen. With respect to the ambiguity engineered by the MD, we add a twist to the agents' preferences by assuming that they are ambiguity averse, and moreover, follow the maxmin expected utility. Now we show that under the ambiguity aversion assumption, the ambiguous mechanism can lead to collusion-proof FSE.

[4] Our prior corresponds to the "weak correlation case" in Laffont and Martimort (2000), where it is shown that there does not exist any mechanism to fulfill collusion-proof FSE. Their argument considers both symmetric mechanisms and asymmetric ones (in their appendix). We refer readers to their Proposition 5 for the proof.

(η_1, η_2)	θ_2^H	θ_2^L
θ_1^H	$(-1.4, -1.4)$	$(-1.9, 0.6)$
θ_1^L	$(0.6, -1.9)$	$(-0.4, -0.4)$

(a) η

(ϕ_1, ϕ_2)	θ_2^H	θ_2^L
θ_1^H	$(2, -2)$	$(-3, 3)$
θ_1^L	$(-3, 3)$	$(2, -2)$

(b) ϕ

Table 3: The η and ϕ parts of transfer rules $\hat{t}$ and $\hat{\hat{t}}$

It is important to notice that from each potential transfer rule $t \in \{\hat{t}, \hat{\hat{t}}\}$, the MD's ex-post payoff, $-0.5q(\theta)^2 - \sum_{i\in I} t_i(\theta)$, is constant across $\theta \in \Theta$ and equal to 0.8. Hence, even if agents manage to jointly misreport via certain collusive mechanism, as the MD's ex-post payoff is constant and equal to the full surplus, collusion cannot dissolve FSE. Hence, it is trivial to verify that this ambiguous mechanism cannot be hurt by collusion. Unfortunately, η, which is the key element contributing to the constant ex-post payoff, does not satisfy the IC constraints by itself, which is why we augment it with $\lambda\phi$ and $-\lambda\phi$.

To verify the feasibility of the ambiguous mechanism, i.e., its IC and IR constraints, we only demonstrate the verification process for type θ_1^H here. For this agent, truthful revelation leads to an interim payoff

$$\min\{0.6(2-1.4+2\lambda)+0.4(1-1.9-3\lambda), 0.6(2-1.4-2\lambda)+0.4(1-1.9+3\lambda)\} = \min\{0,0\} = 0,$$

which means that the IR constraint of θ_1^H binds. Suppose instead that this agent misreports θ_1^L with probability $\gamma \in (0, 1]$. His maxmin expected utility becomes weakly lower:

$$\begin{aligned} &\min\{\gamma[0.6(1 + 0.6 - 3\lambda) + 0.4(0 - 0.4 + 2\lambda)], \gamma[0.6(1 + 0.6 + 3\lambda) + 0.4(0 - 0.4 - 2\lambda)]\} \\ =&\min\{\gamma(0.8 - \lambda), \gamma(0.8 + \lambda)\} \leqslant 0, \end{aligned}$$

where the last inequality follows from the fact that $\lambda \geqslant 0.8$. As a result, the IC constraint of θ_1^H is satisfied.

The construction is such that the interim payoff for a type to truthfully report is independent of the transfer rule $t \in \{\hat{t}, \hat{\hat{t}}\}$, but that for a type to misreport depends on the transfer rule. For types θ_1^H and θ_1^L, the worst-case interim payoff of misreporting is attained by $\hat{t}$, but for types θ_2^H and θ_2^L, the worst-case interim payoff of misreporting is attained by $\hat{\hat{t}}$. The engineered ambiguity reduces the incentive of all types to misreport.

In later sections with more agents, we follow the spirit of the example to achieve collusion-proof FSE. To prevent the grand coalition from manipulation, we make the MD's ex-post payoff constant. To prevent individuals from misreporting, we exploit ambiguity engineered in mechanism rules as well as the assumption of ambiguity aversion. With more agents, there are non-grand coalitions which are not present in the current example. As will be seen, these non-grand coalitions' truth-telling incentives are guaranteed by ambiguity in mechanism rules, but the argument involves additional complication.

We now discuss the important role played by the ambiguity aversion assumption in this example. If agents assign probability ρ and $1-\rho$ to the events that $\hat{t}$ and $\hat{\hat{t}}$ are chosen, then the set of all possible beliefs on the set of transfer rules is $\{(\rho, 1-\rho) : \rho \in [0,1]\}$. The above extreme format of maxmin expected utility corresponds to this case. However, this extreme format of maxmin expected utility is not necessary to achieve collusion-proof FSE. For instance, we can follow Bose and Daripa (2009) in assuming that the MD knows that agents' belief on the true transfer rule deviates from the uniform distribution by only an $\epsilon \in (0,1]$ amount (with $\epsilon = 1$ being the extreme format and $\epsilon \to 0$ being a no ambiguity case). Then the set of multiple beliefs is $\{(1-\epsilon)(0.5,0.5)+\epsilon(\rho,1-\rho) : \rho \in [0,1]\}$. Given this multiple-belief set, the maxmin expected utility of θ_1^H to deviate with probability $\gamma \in (0,1]$ is

$$\begin{aligned}&\min_{\rho\in[0,1]} ((1-\epsilon)0.5+\epsilon\rho)\gamma(0.8-\lambda) + ((1-\epsilon)0.5+\epsilon(1-\rho))\gamma(0.8+\lambda)\\ =&\min_{\rho\in[0,1]} 0.8\gamma + \gamma\lambda\epsilon(1-2\rho) = \gamma(0.8-\lambda\epsilon).\end{aligned}$$

This value can be made nonpositive if the MD chooses a large multiplier $\lambda \geqslant 0.8/\epsilon$ instead. In this case, we can restore collusion-proof FSE. Crucially, the lower bound of λ is not well-defined when $\epsilon = 0$, i.e., when agents are ambiguity neutral. If agents are ambiguity neutral, i.e., assign subjective belief to potential transfers, then the ambiguous mechanism essentially reduces to a standard Bayesian mechanism which is a convex combination of $\hat{t}$ and $\hat{\hat{t}}$. As is argued earlier, standard Bayesian mechanisms cannot achieve collusion-proof FSE. Hence, a positive amount of ambiguity aversion is necessary for our new approach to work.[5]

[5] In this example, the construction also works beyond the MEU preference. For instance, assume that agents face extreme ambiguity but follow the α-maxmin expected utility of Ghirardato and Marinacci (2002), i.e., assign weight $\alpha > 0.5$ to the worst-case transfer rule and $1-\alpha < 0.5$ to the best case. If the MD knows

4 Standard Bayesian Mechanisms

4.1 Definitions

In the motivating example in Section 3, there is a simple way for agents to collude. However, the collusion process can be more complicated. We follow the literature that adopts the mechanism design approach to model information transmission within a coalition, e.g., Laffont and Martimort (2000) and Che and Kim (2006). As we mentioned in Section 1, there is another strand of the literature that imposes the collusion-proofness requirement on the mechanism without explicitly modeling strategic interactions due to information asymmetry within the coalition. In the Online Appendix, we study such an alternative collusion-proofness notion and its implications on FSE.

Now we review the collusion-proofness notion introduced by Che and Kim (2006) (Section 8 therein) and then impose it on FSE mechanisms. We consider the situation where any subset of agents, rather than merely the grand coalition, can collude.

In particular, agents in a coalition S can collude by jointly manipulating their reports to the MD, reallocating the outcome specified by the mechanism (q, t) (e.g., reallocating the object assigned by an auction within the winning coalition), and making further side transfers among themselves. The timing of the mechanism and the collusion process among S is given as follows:

- At date -1, each agent i learns his θ_i.
- At date 0, the MD proposes the mechanism (q, t).

the value of α, then she can also scale up the multiplier λ so that IC is guaranteed. To see this, we revisit the IC constraint of θ_1^H. Under truthful revelation, the α-maxmin expected utility of θ_1^H is equal to 0, but misreporting with probability γ leads to the α-maxmin expected utility of

$$\alpha \min\{\gamma(0.8 - \lambda), \gamma(0.8 + \lambda)\} + (1 - \alpha) \max\{\gamma(0.8 - \lambda), \gamma(0.8 + \lambda)\} = \gamma(0.8 + (1 - 2\alpha)\lambda),$$

which is nonpositive if the MD chooses a large multiplier $\lambda \geqslant \frac{0.8}{2\alpha-1}$. It is worth noting that for the ambiguity neutral case ($\alpha = 0.5$), the lower bound on the right-hand side is not well-defined, and this ambiguous mechanism is perceived as a standard Bayesian mechanism that is a convex combination of $\hat{t}$ and $\hat{\hat{t}}$ with equal weights.

- At date 1, each agent either accepts or rejects the mechanism (q, t).
- At date $1\frac{1}{4}$, a mediator proposes an S-side contract to coalition S.
- At date $1\frac{1}{2}$, each agent in S accepts or rejects the S-side contract.
- At date $1\frac{3}{4}$, if all agents in S accept the S-side contract, then it prescribes the strategy of coalition members at date 2 and the side transfer is realized. Otherwise, no collusion occurs and the agents proceed to date 2 noncooperatively.
- At date 2, if all agents accept (q, t), then it is played; in addition, reallocation within the coalition is realized. Otherwise, agents get their reservation utilities 0.

Specifically, at date $1\frac{1}{4}$, a mediator benevolent to a coalition S can secretly approach S and offer an S**-side contract** (δ^S, ψ^S), which consists of a joint reporting strategy $\delta^S : \Theta_S \to \Delta(\Theta_S)$ and a **side transfer rule**, i.e., a mapping $\psi^S : \Theta_S \to \mathbb{R}^{|S|}$ such that $\sum_{i \in S} \psi_i^S(\theta_S) = 0$ for all $\theta_S \in \Theta_S$.

Following Che and Kim (2006), we focus on the outcome implemented as a result of the S-side contract. Given (q, t) and S, we say $(\tilde{q} : \Theta \to A, \tilde{t} : \Theta \to \mathbb{R}^n)$ is an S**-reallocational manipulation**, if there exists an S**-side contract** (δ^S, ψ^S), such that for all $\theta \in \Theta$,

$$\tilde{t}_i(\theta) = \begin{cases} \sum_{\theta'_S \in \Theta_S} t_i(\theta'_S, \theta_{-S}) \delta^S[\theta_S](\theta'_S) + \psi_i^S(\theta_S) & \text{if } i \in S, \quad (1) \\ \sum_{\theta'_S \in \Theta_S} t_i(\theta'_S, \theta_{-S}) \delta^S[\theta_S](\theta'_S) & \text{otherwise,} \quad (2) \end{cases}$$

$$u_0(\tilde{q}(\theta)) = \sum_{\theta'_S \in \Theta_S} u_0\big(q(\theta'_S, \theta_{-S})\big) \delta^S[\theta_S](\theta'_S), \tag{3}$$

$$u_i(\tilde{q}(\theta), \theta) = \sum_{\theta'_S \in \Theta_S} u_i\big(q(\theta'_S, \theta_{-S}), \theta\big) \delta^S[\theta_S](\theta'_S), \forall i \notin S. \tag{4}$$

By (1) and (2), after the coalitional manipulation, agents in S receive monetary transfers according to the manipulated reports as well as side transfers, but agents out of S only receive the former. By (3) and (4), the reallocation can only take place within S and thus should be undetectable by the MD and the noncollusive agents (agents out of S). If equation (4) also holds for all $i \in S$, then $(\tilde{q}, \tilde{t})$ reduces to an S**-communicative manipulation**, which does not involve reallocation of the outcome. Given (q,t), we may denote $(\tilde{q},\tilde{t})$, an S-reallocational manipulation (resp. S-communicative manipulation) induced by S-side contract (δ^S,ψ^S), as $(\tilde{q},t^{\delta^S}+\psi^S)$ (resp. $(q^{\delta^S},t^{\delta^S}+\psi^S)$) to highlight its structure. The S-reallocational manipulation $(\tilde{q},\tilde{t})$ is said to be S**-feasible**, if no agent in S has the incentive to decline or misreport in $(\tilde{q},\tilde{t})$, i.e.,

$$
\begin{aligned}
&S\text{-IR} \qquad V_i[\tilde{q},\tilde{t}](\theta_i,\bar{\sigma}_i) \geqslant V_i[q,t](\theta_i,\bar{\sigma}_i), \forall i \in S, \theta_i \in \Theta_i,\\
&S\text{-IC} \qquad V_i[\tilde{q},\tilde{t}](\theta_i,\bar{\sigma}_i) \geqslant V_i[\tilde{q},\tilde{t}](\theta_i,\sigma_i), \forall i \in S, \theta_i \in \Theta_i, \sigma_i : \Theta_i \to \Delta(\Theta_i).
\end{aligned}
$$

Although the above S-IR constraint is an interim one, i.e., is stated for each $\theta_i \in \Theta_i$, a weighted sum of the above S-IR constraints across all $i \in S$ and $\theta_i \in \Theta_i$ implies that

$$
\begin{aligned}
\sum_{\theta_S \in \Theta_S} V_S[\tilde{q},\tilde{t}](\theta_S,\bar{\delta}_S)p(\theta_S) &= \sum_{i\in S}\sum_{\theta_i\in\Theta_i} V_i[\tilde{q},\tilde{t}](\theta_i,\bar{\sigma}_i)p(\theta_i)\\
\overset{S\text{-IR}}{\geqslant} \sum_{i\in S}\sum_{\theta_i\in\Theta_i} V_i[q,t](\theta_i,\bar{\sigma}_i)p(\theta_i) &= \sum_{\theta_S\in\Theta_S} V_S[q,t](\theta_S,\bar{\delta}^S)p(\theta_S).
\end{aligned}
$$

Namely, engaging in collusion does not hurt S in terms of members' aggregate ex-ante utility.

In the collusion-proofness notion adopted by Che and Kim (2006), only one coalition, denoted by S, may be formed. Agents can collude via any S-feasible reallocational manipulation, but if a collusion takes place, it should neither hurt the MD nor lead to an infeasible mechanism. In particular, if a manipulation of S is anticipated but violates noncollusive agents' IR or IC constraints, then the noncollusive agents will decline or misreport in the main mechanism, which eventually jeopardizes the MD's payoff.

Formally, for any $S \in 2^I\backslash\{\varnothing\}$ with $|S| \geqslant 2$, a feasible mechanism (q,t) is said to satisfy the robust collusion-proofness condition with respect to S (RCP with respect to S), if every S-feasible S-reallocational manipulation $(\tilde{q},\tilde{t})$ is a feasible mechanism and satisfies

$$
\sum_{\theta\in\Theta}[u_0\big(q(\theta)\big) - \sum_{i\in I} t_i(\theta)]p(\theta) = \sum_{\theta\in\Theta}[u_0\big(\tilde{q}(\theta)\big) - \sum_{i\in I}\tilde{t}_i(\theta)]p(\theta).
$$

In the current paper, we assume that the MD is concerned about joint manipulations of all possible coalitions, rather than just one. Hence, our collusion-proofness notion is more demanding than requiring RCP with respect to any particular coalition. A feasible mechanism (q,t) is said to satisfy the **robust collusion-proofness** condition (RCP) if it satisfies RCP with respect to every $S \in 2^I\backslash\{\varnothing\}$ with $|S| \geqslant 2$.

4.2 Result

In a two-agent two-type framework, Laffont and Martimort (2000) imply that the MD may fail to obtain FSE via a standard Bayesian mechanism that satisfies RCP with respect to the only coalition, i.e., the grand one.[6] However, according to Che and Kim (2006), in a framework with at least three agents, for any finite type space and almost every prior over it, the corresponding information structure can guarantee FSE via mechanisms satisfying RCP with respect to the grand coalition.

In this section, Proposition 1 shows that the RCP condition, or the unknown makeup of the colluding coalition, drastically restricts the information structures that guarantee FSE. This result extends Laffont and Martimort (2000)'s negative message on collusion-proof FSE to environments with more agents and more types, and significantly weakens the positive result of Che and Kim (2006). Hence, collusion of unknown coalitions is one potential resolution of Crémer and McLean (1985, 1988)'s paradox.

Proposition 1. *Suppose* Θ *is such that* $n \geqslant 4$ *and* $|\Theta| \geqslant 24$*, and that there exists* $j \in I$ *for whom* $|\Theta_j| \leqslant |\Theta_{-j}|$ *and* $|\Theta_i| \leqslant |\Theta_{-j-i}|$ *for all* $i \in I \backslash \{j\}$*. Under almost every prior* $p \in \Delta(\Theta)$*, the information structure* (Θ, p) *cannot guarantee FSE via mechanisms satisfying the RCP condition.*

In Proposition 1, the dimensional restrictions are satisfied under the most commonly studied case where agents' type sets are equally large and when the number of agents is large enough: when $n \geqslant 5$, or $n \geqslant 4$ and each agent has at least three types. Also, notice that Proposition 1 provides a sufficient condition on the information structure such that collusion-proof FSE cannot be guaranteed. This condition is not necessary: there are other information structures that cannot guarantee collusion-proof FSE. Recall the two-agent public good example of Laffont and Martimort (2000) discussed in Section 3.

At last, we provide a sketch of the proof and leave the details to Appendix A.2. This

[6] In fact, the collusion-proofness notion of Laffont and Martimort (2000) is called "weak collusion-proofness". Che and Kim (2006) have shown that when a mechanism extracts the full surplus and thus involves an efficient allocation rule, the requirement of RCP with respect to I is stronger than weak collusion-proofness.

proof follows a duality approach and is different from the argument of Laffont and Martimort (2000).

We first establish the following result irrespective of the dimensionality of Θ.

Lemma 1. *Under any information structure* (Θ, p)*, there exists a payoff structure* $(u_0, (u_i)_{i\in I})$*, such that for any FSE standard Bayesian mechanism* (q,t)*, there exists* $S \in \{I\backslash\{n\}, I\}$ *and two type profiles* $\theta_S \neq \theta'_S \in \Theta_S$ *such that* $V_S[q,t](\theta_S, \theta'_S) > V_S[q,t](\theta_S, \theta_S)$*.*

Then we fix an FSE mechanism (q,t) and one S such that the above inequality holds. Let $\hat{\delta}^S : \Theta_S \to \Delta(\Theta_S)$ be a joint reporting strategy that maximizes the ex-ante total utility of all agents in S, which cannot be the truthful reporting strategy.

Now consider a sub-environment with agents in S only. Let $(\Theta_S, \hat{p})$ be the information structure, where $\hat{p}$ is the marginal distribution of p over Θ_S. Under our dimensionality restriction, almost every $p \in \Delta(\Theta)$ leads to $\hat{p}$ satisfying the Convex Independence condition and the Identifiability condition (Definition 3 in Appendix A.1) in the sub-environment. Crucially, we show the existence of a side transfer rule ψ^S such that $(q^{\hat{\delta}^S}, t^{\hat{\delta}^S} + \psi^S)$ is an S-feasible S-communicative manipulation. This is done by adapting the main result of Kosenok and Severinov (2008) to a sub-environment with agents in S only: there exists an ex-post budget balanced $\psi^S : \Theta_S \to \mathbb{R}^{|S|}$ such that it is part of a feasible mechanism. The feasibility is then used to establish the S-feasibility of the S-communicative manipulation in the original environment. Since q is efficient, the S-communicative manipulation either decreases the ex-ante payoff of the MD, or hurts at least one agent out of S. In either case, (q,t) does not satisfy RCP with respect to S, and thus does not satisfy RCP.

5 Ambiguous Mechanisms

5.1 Definitions

Section 4 has shown that it may be difficult to achieve FSE via standard Bayesian mechanisms satisfying the RCP condition. To resolve this problem, we follow the motivating example and allow the MD to consider a broader set of tools called ambiguous mechanisms (see, e.g., Bose and Renou, 2014; Di Tillio et al., 2017; Guo, 2019; Tang and Zhang, 2021) and assume that agents are ambiguity averse with respect to the engineered ambiguity.

An **ambiguous mechanism** is a publicly announced nonempty compact set of mechanisms, among which agents do not know the true mechanism that the MD has committed to.[7] Other than the ambiguity engineered by the MD, we assume that there is no other exogenous ambiguity and agents still share an unambiguous common prior p. This simplifying assumption helps us to compare the results in the current section with those from the previous section, and thus highlight the freedom granted by ambiguous mechanisms. As we will show in this section, focusing on direct mechanisms and ambiguity in transfer rules alone is sufficient to turn the technically unpermissive result on collusion-proof FSE in Section 4 into a possibility result.[8] Hence, for FSE, we focus on the case where all mechanisms share an efficient allocation rule q, and T is the set of potential transfer rules. Formally, given $(u_0, (u_i)_{i\in I})$, we say an ambiguous mechanism (q, T) achieves FSE, if it is feasible (remains to be redefined) and

$$\sum_{\theta\in\Theta}[u_0\big(q(\theta)\big) - \sum_{i\in I} t_i(\theta)]p(\theta) = FS, \forall t \in T. \tag{5}$$

When the set T is a singleton, the ambiguous mechanism reduces to a standard Bayesian mechanism that we have been using in previous sections.

For simplicity, we assume that each agent uses a special format of the maxmin expected utility (MEU) of Gilboa and Schmeidler (1989): in particular, an agent makes decision with the worst-case mechanism only. Formally, assume that an agent $i \in I$ with type $\theta_i \in \Theta_i$ holds a nonempty, compact, and convex set of probabilistic assessments over $T \times \Theta_{-i}$: the set of all distributions over $T \times \Theta_{-i}$ whose marginal distribution over Θ_{-i} is $p(\cdot|\theta_i)$. In the multiple-belief set, the worst-case expected utility of following strategy $\sigma_i : \Theta_i \to \Delta(\Theta_i)$ is $V_i[q, T](\theta_i, \sigma_i) \equiv \min_{t\in T} V_i[q, t](\theta_i, \sigma_i)$. When T is a singleton, the MEU is consistent with the subjective expected utility. By replacing $V_i[q, t](\theta_i, \sigma_i)$ with $V_i[q, T](\theta_i, \sigma_i)$, we can redefine

[7] We impose the compactness assumption so that the minimization operator can be conveniently used.

[8] Bose and Renou (2014) have introduced an indirect mechanism to engineer ambiguous beliefs. When ambiguous beliefs are present, Renou (2015) and Lopomo et al. (2020) show that it is no longer generically possible to guarantee FSE. Whether inducing ambiguous beliefs and ambiguous mechanisms simultaneously can further help the MD in achieving FSE (or collusion-proof FSE) remains an open question.

the IR, IC, and the feasibility conditions. As is discussed in Section 3, the usefulness of ambiguous mechanisms in overturning the negative result in Section 4 does not necessarily require the extreme format of MEU, but requires strict ambiguity aversion. If agents are ambiguity neutral instead, even if an ambiguous mechanism (q, T) is provided, agents would reduce (q, T) to a standard Bayesian mechanism (q, t) where t is a convex combination of transfer rules in T. In this case, the unpermissive result of Proposition 1 also applies to the ambiguity neutrality scenario.

It is natural to assume that a coalition S can also collude via an ambiguous S-side contract. We will discuss later the case where each potential side transfer rule takes a similar form as the one in Section 4, i.e., requires an input $\theta_S \in \Theta_S$ only. For now, we generalize each potential side transfer rule by allowing it to depend on $\theta_S \in \Theta_S$, $\theta_{-S} \in \Theta_{-S}$, and $t \in T$, where the latter two are revealed from the main mechanism. This makes the set of ambiguous S-side contract (weakly) broader and thus makes collusion (weakly) easier. An **ambiguous *S*-side contract** is a pair (δ^S, Ψ^S), where Ψ^S is a set of potential side transfer rules and each element in it, $\psi^S : T \times \Theta \to \mathbb{R}^{|S|}$, is required to satisfy $\sum_{i \in S} \psi_i^S(t, \theta) = 0$ for all $t \in T$ and $\theta \in \Theta$.

The timing of the ambiguous mechanism and the ambiguous S-side contract is almost identical with that described in Section 4 except for minor differences at dates $1\frac{3}{4}$ and 2.

- At date $1\frac{3}{4}$, if all agents in S accept (δ^S, Ψ^S), then δ^S prescribes the strategy of coalition members at date 2 and the mediator uncovers the side transfer rule that she secretly commits to. Otherwise, no collusion occurs and the agents proceed to date 2 noncooperatively.

- At date 2, if all agents accept (q, T), they report to the mechanism (q, T), and then the MD uncovers the true transfer rule $t \in T$ that she secretly commits to as well as the types reported to (q, T); allocation and transfers are realized according to q and $t \in T$; in addition, reallocation and the side transfer within S are realized. Otherwise, agents get their reservation utilities 0.

Relative to the timing described in Section 4, we postpone the payment of the side transfer in this section to date 2 after $\theta_{-S} \in \Theta_{-S}$ and $t \in T$ are realized from the main mechanism.

Given (q, T) and coalition S, we say $(\tilde{q}, \tilde{T})$ is an **ambiguous *S*-reallocational manipulation**, if there exists an ambiguous S-side contract (δ^S, Ψ^S) such that (i) for each $\tilde{t} \in \tilde{T}$, there exists $t \in T$ and $\psi^S \in \Psi^S$ such that for all $\theta \in \Theta$,

$$\tilde{t}_i(\theta) = \sum_{\theta'_S \in \Theta_S} t_i(\theta'_S, \theta_{-S})\delta^S[\theta_S](\theta'_S) + \psi_i^S(t, \theta) \text{ if } i \in S, \tag{6}$$

and (2), (3), and (4) are satisfied, and (ii) for each $t \in T$ and $\psi^S \in \Psi^S$, the transfer rule $\tilde{t}$ defined by (6) and (2) is an element of $\tilde{T}$. From this ambiguous S-reallocational manipulation, we assume that type-θ_i agent i's MEU from following σ_i takes the form

$$V_i[\tilde{q}, \tilde{T}](\theta_i, \sigma_i) = \begin{cases} \min_{\psi^S \in \Psi^S, t \in T} \sum_{\hat{\theta}_i \in \Theta_i} \left[V_i[\tilde{q}, t^{\delta^S}](\theta_i, \hat{\theta}_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \psi_i^S(t, \hat{\theta}_i, \theta_{-i}) p(\theta_{-i}|\theta_i) \right] \sigma_i[\theta_i](\hat{\theta}_i), & \text{if } i \in S, \\ \min_{t \in T} \sum_{\hat{\theta}_i \in \Theta_i} V_i[\tilde{q}, t^{\delta^S}](\theta_i, \hat{\theta}_i) \sigma_i[\theta_i](\hat{\theta}_i), & \text{if } i \notin S. \end{cases}$$

We may also denote this ambiguous S-reallocational manipulation by $(\tilde{q}, T^{\delta^S} + \Psi^S)$ to highlight its structure.

By replacing every $V_i[\tilde{q}, \tilde{t}](\theta_i, \sigma_i)$ in Section 4 with $V_i[\tilde{q}, \tilde{T}](\theta_i, \sigma_i)$, we can redefine the S-feasibility condition for an ambiguous S-reallocational manipulation and the **robust collusion-proofness*** (RCP*) condition for an ambiguous mechanism. If we do not allow $\theta_{-S} \in \Theta_{-S}$ and $t \in T$ to be part of the inputs of each $\psi^S \in \Psi^S$, as in Section 4, then the RCP* condition reduces to the **robust collusion-proofness** (RCP) condition under ambiguous mechanisms.

5.2 Result

Do ambiguous mechanisms help the MD guarantee FSE via collusion-proof mechanisms? If yes, to what extent? To answer these questions, we first adapt the Beliefs Determine Preferences property of Neeman (2004) and introduce a definition.

Definition 1. *1. For $S \in 2^I \backslash \{\varnothing, I\}$, the prior p satisfies the S-**Beliefs Determine Preferences** (S-BDP) property, if $p(\cdot|\theta_S) \neq p(\cdot|\theta'_S)$ for each pair of $\theta_S \neq \theta'_S \in \Theta_S$.*

*2. The prior p satisfies the **Beliefs Determine Preferences** (BDP) property, if it satisfies the S-BDP property for all singleton $S \in 2^I$.*

3. *The prior p satisfies the* ***Coalition Beliefs Determine Preferences*** *(CBDP) property, if it satisfies the S-BDP property for all $S \in 2^I \backslash \{\varnothing, I\}$.*

The CBDP property implies that knowing the posterior belief of some $S \in 2^I \backslash \{\varnothing, I\}$ over types of agents in $I \backslash S$ can uniquely identify the type profile of S. By definition, the CBDP property implies the BDP property.[9] When agents have independent beliefs, the prior neither satisfies the BDP property nor the CBDP property. However, the BDP and CBDP properties impose a weak restriction on priors over a fixed finite-dimensional Θ: among all priors over Θ, the ones for which the CBDP property fails constitute a set of measure zero.[10]

Proposition 2 shows that the CBDP property characterizes information structures that guarantee the existence of an FSE ambiguous mechanism satisfying the RCP* condition.

Proposition 2. *Given an information structure (Θ, p), the following statements are equivalent:*

1. *The CBDP property holds for prior p.*

2. *The information structure (Θ, p) guarantees FSE via ambiguous mechanisms satisfying the RCP* condition.*

As the RCP* condition is stronger than the RCP condition, we immediately have the following result.

Corollary 1. *Given an information structure (Θ, p), if the CBDP property holds for prior p, then the information structure (Θ, p) guarantees FSE via ambiguous mechanisms satisfying the RCP condition.*

---

[9] To see that the CBDP property can be strictly stronger, consider an example with $I = \{1, 2, 3\}$ and $\Theta_i = \{\theta_i^1, \theta_i^2\}$ for each $i \in I$. We collapse the agent index and denote, for instance, $(\theta_1^1, \theta_2^2, \theta_3^1)$ by θ^{121}. The following prior p satisfies the BDP property, where $p(\theta^{111}) = 0.1$, $p(\theta^{112}) = 0.2$, $p(\theta^{121}) = 0.1$, $p(\theta^{122}) = 0.1$, $p(\theta^{211}) = 0.1$, $p(\theta^{212}) = 0.1$, $p(\theta^{221}) = 0.2$, and $p(\theta^{222}) = 0.1$. However, $p(\cdot|\theta_1^1, \theta_2^2) = p(\cdot|\theta_1^2, \theta_2^1) = (0.5, 0.5)$, and thus, p does not satisfy the CBDP property.

[10] Following Crémer and McLean (1988), Che and Kim (2006), and Kosenok and Severinov (2008), the current paper focuses on a fixed finite type space to study mechanism design with correlated beliefs. Without fixing the dimension of the type space a priori, there are works (see, e.g., Heifetz and Neeman, 2006; Chen and Xiong, 2013) discussing how generic the BDP property is.

A sketch of the proof to Proposition 2 is provided below and details are relegated to Appendix A.3.

To establish Statement 2 $\Rightarrow$ Statement 1, we begin with any information structure where the prior does not satisfy the CBDP property. When the prior does not satisfy the BDP property, we construct a payoff structure for which there exists no FSE ambiguous mechanism. When the prior satisfies the BDP property but violates the CBDP property, we can construct a payoff structure, so that for every FSE ambiguous mechanism, there is an ex-ante strictly profitable joint reporting strategy for a coalition. Taking advantage of the BDP property, Lemma 5 constructs an ambiguous side contract to implement the collusion. The consequent communicative manipulation hurts either the MD or a noncollusive agent, and thus this FSE ambiguous mechanism cannot satisfy the RCP* condition.

To sketch the proof of Statement 1 $\Rightarrow$ Statement 2, we first establish the existence of a transfer rule η, under which the MD's ex-post payoff, $u_0\big(q(\theta)\big) - \sum_{i\in I}\eta_i(\theta)$, is constant and equal to FS, and agents' IR constraints bind. The feature of constant ex-post payoff will be useful in preventing any I-feasible ambiguous I-reallocational manipulation from affecting the MD's payoff. However, η neither addresses the IC constraints, nor the RCP* condition with respect to any non-grand coalition S. Therefore, we adjust η with two groups of transfer rules. The following lemmas are helpful.

Lemma 2. *For any coalition $i \in S$, and $\bar{\theta}_i \in \Theta_i$, if there does not exist $\hat{\theta}_i \in \Theta_i\backslash\{\bar{\theta}_i\}$ such that $p(\cdot|\hat{\theta}_i) = p(\cdot|\bar{\theta}_i)$, then there exists an ex-post budget balanced transfer rule $\phi^{\bar{\theta}_i} : \Theta \to \mathbb{R}^n$ such that*

(i) $\sum_{\theta_{-j}\in\Theta_{-j}} \phi_j^{\bar{\theta}_i}(\theta_j, \theta_{-j})p(\theta_{-j}|\theta_j) = 0$ *for all $j \in I$ and $\theta_j \in \Theta_j$,*

(ii) $\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^{\bar{\theta}_i}(\hat{\theta}_i, \theta_{-i})p(\theta_{-i}|\bar{\theta}_i) < 0$ *for all $\hat{\theta}_i \in \Theta_i\backslash\{\bar{\theta}_i\}$.*

Lemma 2 is an immediate result of Lemma 4 in Appendix A.1. Hence, we omit the proof of Lemma 2. It is useful in constructing an ambiguous mechanism that overcomes type-$\bar{\theta}_i$ agent i's incentive to misreport.

Lemma 3. *Fix any $S \in 2^I$ with $2 \leqslant |S| \leqslant n-1$. If the S-BDP property holds, then there exists an ex-post budget balanced transfer rule $\phi^S : \Theta \to \mathbb{R}^n$ such that* *(i)* $\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^S(\theta_i, \theta_{-i})p(\theta_{-i}|\theta_i) = 0$ *for all* $i \in I$ *and* $\theta_i \in \Theta_i$*;*

(ii) $\sum_{\hat{\theta}_S\in\Theta_S} \sum_{\bar{\theta}_S\in\Theta_S} \sum_{i\in S} \sum_{\theta_{-S}\in\Theta_{-S}} \phi_i^S(\hat{\theta}_S, \theta_{-S})p(\bar{\theta}_S, \theta_{-S})\delta^S[\bar{\theta}_S](\hat{\theta}_S) < 0$ *for all non-truthful deterministic* $\delta^S \neq \bar{\delta}^S$*.*

When $n \geqslant 3$, Lemma 3 is helpful to construct an ambiguous mechanism that is immune from collusion of the non-grand coalition S. Note that Lemma 3 imposes the equality constraints on individual agents at the interim stage (i.e., at each $\theta_i \in \Theta_i$), similar to Lemma 2. However, Lemma 3 imposes the inequality constraint at the ex-ante stage for the coalition S as a whole.

When the CBDP property holds, define $T = \{\eta + \lambda_1\phi^{\tilde{\theta}_i}|i \in I, \tilde{\theta}_i \in \Theta_i\} \cup \{\eta + \lambda_2\phi^S|S \in 2^I\backslash\{\varnothing, I\} \text{ with } 2 \leqslant |S| \leqslant n-1\}$, where λ_1 and λ_2 are two large numbers.[11]

The IR constraints bind in this ambiguous mechanism, because η is constructed such that IR constraints bind, and because of requirement (i) in Lemma 2 and requirement (i) in Lemma 3. Notice that that no ambiguity is perceived by any type-θ_i agent i when all agents truthfully report, because under each $t \in T$, the interim payoff of θ_i is zero.

It is important to notice that this ambiguous mechanism gives the MD a constant ex-post payoff that is equal to FS under each $t \in T$, because η is constructed in such a way, and because of the ex-post budget balance of each $\phi^{\tilde{\theta}_i}$ and ϕ^S. Namely, our ambiguous mechanism is a full-insurance mechanism from the MD's perspective. As a result, no I-feasible ambiguous I-reallocational manipulation can affect the MD's payoff, which establishes the RCP* condition with respect to I.

The set $\{\eta + \lambda_1\phi^{\tilde{\theta}_i}|i \in I, \tilde{\theta}_i \in \Theta_i\}$ makes sure that agents have no incentive to misreport unilaterally. To see this, by (ii) in Lemma 2, any unilateral deviation earns type-$\tilde{\theta}_i$ agent i a negative expected transfer under $\phi^{\tilde{\theta}_i}$. When the multiplier λ_1 is sufficiently large, $\eta + \lambda_1\phi^{\tilde{\theta}_i} \in T$ gives $\tilde{\theta}_i$ a negative expected utility, which bounds his MEU of misreporting from above and eventually establishes the IC condition.

Suppose a mediator secretly approaches coalition S with $2 \leqslant |S| \leqslant n-1$ and proposes an ambiguous S-side contract that involves misreporting. Under transfer rule $t = \eta + \lambda_2\phi^S \in T$, the total utility of coalition S under the manipulation is negative in the ex-ante stage. This

[11]When $n = 2$, $T = \{\eta + \lambda_1\phi^{\tilde{\theta}_i}|i \in I, \tilde{\theta}_i \in \Theta_i\}$ satisfies the RCP* condition.

implies that no ambiguous S-side contract involving misreporting can lead to an S-IR, and a fortiori an S-feasible, ambiguous S-reallocational manipulation. Hence, (q, T) satisfies the RCP* condition with respect to S.

5.3 Discussions

We now discuss the connection of our Proposition 2 with the literature in detail.

If we view the results of Crémer and McLean (1985, 1988) as a paradox, our Proposition 1 shows that collusion can be one resolution of the paradox, but Proposition 2 shows that the use of ambiguous mechanisms can restore the paradox.

In particular, in the two-agent setup, Proposition 2 can soften the theoretically unpermissive result of Laffont and Martimort (2000) on collusion-proof FSE. Notice that with only two agents, the CBDP property is equivalent to the BDP property, under which we can guarantee collusion-proof FSE.

Our Proposition 2 is also related to Theorem 2 and Corollary 2 of Che and Kim (2006), where it is shown that if the prior satisfies Convex Independence and their Condition PI', then the corresponding information structure can guarantee FSE via standard Bayesian mechanisms that satisfy the RCP condition with respect to I. Our CBDP property is neither stronger nor weaker than their sufficient conditions. However, recall that our RCP* condition is stronger than their collusion-proofness notion (RCP with respect to I), partly because we require the mechanism to be immune from all coalitions' manipulations, and partly because we allow side contracts to address more contingencies. In fact, when aiming to design an FSE ambiguous mechanism that is RCP with respect to I only, we can modify the current proofs and show that the BDP property, which is weaker than Convex Independence, is sufficient. Namely, ambiguous mechanisms are more potent in achieving collusion-proof FSE in the sense of Che and Kim (2006), even when a coalition can use ambiguous side contracts to combat the ambiguity in the main mechanism. Moreover, our paper provides a construction of an FSE (ambiguous) mechanism that satisfies the RCP/RCP* condition with respect to any non-grand coalition S, but the counterpart under Bayesian mechanisms has not been explicitly studied. Che and Kim (2006) provide a Bayesian mechanism that satisfies the RCP condition with respect to one known non-grand coalition S under independent beliefs, but not under correlated beliefs which are necessary for guaranteeing FSE.

The current paper also extends the approach of Guo (2019), where it is shown that FSE can be guaranteed under ambiguous mechanisms if and only if the prior satisfies the BDP property. The leading distinction is that the current paper focuses on collusion concerns, which bring new challenges in constructing a full-insurance transfer rule η, preventing an ex-ante profitable deviation in Lemma 3, and designing the ambiguous side contract in Lemma 5, etc. In addition, we remark that the earlier paper, as well as many works on mechanism design under ambiguity, only focuses on pure strategy deviations, which may be with loss of generality because mixed strategies sometimes can be played to hedge against ambiguity.[12] However, the current paper explicitly addresses misreporting in mixed strategies, and thus, the concern of hedging does not apply.

For simplicity, the paper has been assuming that agents only consider the worst-case mechanism. This assumption is a bit extreme, but there are less extreme MEU models under which our results at least hold partially. As a brief illustration, we adjust the ambiguous mechanism satisfying the RCP* condition constructed for Proposition 2 into a "symmetric" one so that it works under the MEU model with ϵ-contaminated multiple-belief set used in, e.g., Bose and Daripa (2009). Let $T \equiv \{\eta + \lambda_1 \phi^{\tilde{\theta}_i} | i \in I, \tilde{\theta}_i \in \Theta_i\} \cup \{\eta - \lambda_1 \phi^{\tilde{\theta}_i} | i \in I, \tilde{\theta}_i \in \Theta_i\} \cup \{\eta + \lambda_2 \phi^C | C \in 2^I \backslash \{\varnothing, I\} \text{ with } 2 \leqslant |C| \leqslant n-1\} \cup \{\eta - \lambda_2 \phi^C | C \in 2^I \backslash \{\varnothing, I\} \text{ with } 2 \leqslant |C| \leqslant n-1\}$. Suppose an agent's multiple-belief set over transfer rules is the collection of convex combinations between the uniform distribution and any other distribution with $\epsilon > 0$ weight on the latter. In this case, it is easy to enlarge λ_1 and λ_2 used in the proof of Proposition 2 by a factor of $\frac{1}{\epsilon}$ to establish RCP* under the less extreme MEU model. Again, it is worth noting that for the ambiguity neutral case ($\epsilon = 0$), the factor is not well-defined. In this case, ambiguous mechanisms are perceived as standard Bayesian mechanisms, and are not helpful to achieve collusion-proof FSE.[13]

[12] See, for example, Ke and Zhang (2020) for a discussion on randomization at different stages.

[13] The above construction also works for the α-MEU model of Ghirardato and Marinacci (2002) when $n = 2$ and $\alpha > 0.5$, except that we may need to use a different λ_1. How to extend our approach to preferences beyond the family of MEU preferences with $n \geqslant 3$ remains an open question.

6 Concluding Remarks

We end the paper with two open questions.

First, there are alternative collusion-proofness notions of interest. One strand of the literature, e.g., Green and Laffont (1979) and Safronov (2018), views every coalition S as a pseudo agent with type set Θ_S. In Safronov (2018), the pseudo agent has a "utility function" which is the sum of members' utility functions and maximizes the interim utility. The collusion-proofness notion therein requires that no pseudo agent has the incentive to misreport. The alternative notion does not affect the main message of the paper, i.e., it is difficult to guarantee collusion-proof FSE via standard Bayesian mechanisms, but much easier via ambiguous mechanisms. We formalize these results in the Online Appendix.

Moreover, one may consider a variant of the RCP condition where the mediator can only coordinate joint deviations with side contracts that are immune from further deviations of coalitions. This requirement restricts the class of side contracts that a mediator can use and hence, weakens the RCP condition. Whether such an alternative collusion-proofness notion can soften the theoretically unpermissive result in Proposition 1 or relax the necessity of the CBDP property in Proposition 2 remains an open question.

Second, as in Crémer and McLean (1985, 1988), Che and Kim (2006), and Kosenok and Severinov (2008), we focus on properties of the information structure such that collusion-proof FSE can be guaranteed. To establish Proposition 1 and the necessity direction of Proposition 2, we adopt particular payoff structures for which FSE cannot be achieved. This approach does not exclude the possibility that there are payoff structures for which collusion-proof FSE can be achieved. Identifying those payoff structures remains an open question.

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A Appendix

A.1 Preparatory Notations, Definitions, and Results

This paper establishes the existence of a transfer rule satisfying certain constraints by applying (a corollary of) the transposition theorem of Motzkin (1951) or the alternative theorem of Fredholm (1903). We present them and establish a preparatory lemma for constructing an ambiguous mechanism and an ambiguous S-side contract in Lemmas 2 and 5.

Theorem 1 (Motzkin, 1951)**.** *Let $B \in \mathbb{R}^{m\times l}$ and $D \in \mathbb{R}^{k\times l}$ be matrices. Exactly one of the following holds: either the system $Bx < \boldsymbol{0}_{m\times 1}$, $Dx = \boldsymbol{0}_{k\times 1}$ has a column vector solution*

$x \in \mathbb{R}^l$, *or there exist column vectors* $y_1 \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ *and* $y_2 \in \mathbb{R}^k$ *such that* $B'y_1 + D'y_2 = \mathbf{0}_{l\times 1}$.

We remark that $Bx < \mathbf{0}_{m\times 1}$ means that all m strict inequalities must hold.

Theorem 2 (Fredholm, 1903)**.** *Let* $B \in \mathbb{R}^{m\times l}$ *be a matrix and* b *be a column vector in* $\mathbb{R}^m$. *Exactly one of the following holds: either the system* $Bx = b$ *has a column vector solution* $x \in \mathbb{R}^l$, *or* $B'y = \mathbf{0}_{l\times 1}$ *has a column vector solution* $y \in \mathbb{R}^m$ *with* $y'b \neq 0$.

To apply these theorems, it is important to construct matrices B and D. As a preparation, we fix any order of the elements in Θ and define some row vectors in $\mathbb{R}^{n|\Theta|}$. For each $x \in \mathbb{R}^{n|\Theta|}$, divide its dimensions into n blocks of $|\Theta|$ dimensions. Let the first block of $|\Theta|$ dimensions corresponds to agent 1, ..., and the last block corresponds to agent n. Within each block, the dimensions correspond to elements of Θ. Hence, each dimension of $x \in \mathbb{R}^{n|\Theta|}$ corresponds to an agent and a type profile.

For each $S \in 2^I\backslash\{\varnothing\}$, $C \in 2^I\backslash\{\varnothing, I\}$ and type profiles $\theta_C, \theta'_C \in \Theta_C$ (may be identical), we define a row vector $p^S_{\theta_C\theta'_C} \in \mathbb{R}^{n|\Theta|}_+\backslash\{\mathbf{0}\}$ as follows. For each $i \in S$ and $\theta_{-C} \in \Theta_{-C}$, let the dimension of $p^S_{\theta_C\theta'_C}$ corresponding to agent i and type profile (θ'_C, θ_{-C}) be equal to $p(\theta_C, \theta_{-C})$, where p is the prior. Thus, we have defined $|S||\Theta_{-C}|$ dimensions of $p^S_{\theta_C\theta'_C}$. Let all other dimensions of $p^S_{\theta_C\theta'_C}$ be 0.

For each $\theta \in \Theta$ and $S \in 2^I\backslash\{\varnothing\}$, define a row vector $e^S_\theta \in \mathbb{R}^{n|\Theta|}_+\backslash\{\mathbf{0}\}$ as follows. For each $i \in S$, let the dimension of e^S_θ corresponding to i and θ be equal to 1. Thus, we have defined $|S|$ dimensions of e^S_θ. Let all other dimensions of e^S_θ be 0.

For example, let $I = \{1, 2, 3\}$ and $\Theta_i = \{\theta^1_i, \theta^2_i\}$ for each $i \in I$. We order the eight elements of Θ by: θ^{111}, θ^{112}, θ^{121}, θ^{122}, θ^{211}, θ^{212}, θ^{221}, θ^{222}, where for instance, $\theta^{121} \equiv (\theta^1_1, \theta^2_2, \theta^1_3)$. For each vector in $\mathbb{R}^{24}$, its first, second, and third blocks of eight dimensions correspond to agents 1, 2, and 3, respectively. Let $\mathbf{0}_{1\times k}$ denote a zero row vector in $\mathbb{R}^k$. We illustrate with two vectors below and use a box to group every block of eight dimensions:

$$p^{\{3\}}_{(\theta^1_1,\theta^1_2)(\theta^1_1,\theta^2_2)} = (\boxed{\mathbf{0}_{1\times 8}}, \boxed{\mathbf{0}_{1\times 8}}, \boxed{\mathbf{0}_{1\times 2}, p(\theta^{111}), p(\theta^{112}), \mathbf{0}_{1\times 4}});$$
$$e^{\{1,2\}}_{\theta^{112}} = (\boxed{0, 1, \mathbf{0}_{1\times 6}}, \boxed{0, 1, \mathbf{0}_{1\times 6}}, \boxed{\mathbf{0}_{1\times 8}}).$$

Lemma 4 below provides a unified approach to establish two technical observations used in the proof of Lemmas 2 and 5. It establishes the existence of a budget balanced monetary transfer within a coalition S, where S may be I or a proper subset of I. The monetary transfer is contingent on all agents' reported types, rather than those in coalition S only. The transfer rule gives each agent in S zero on-path interim transfer, but gives type-$\bar{\theta}_j$ agent j a negative interim transfer when he unilaterally misreports.

Lemma 4. *For any coalition S, $j \in S$, and $\bar{\theta}_j \in \Theta_j$, if there does not exist $\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}$ such that $p(\cdot|\hat{\theta}_j) = p(\cdot|\bar{\theta}_j)$, then there exists a transfer rule $\xi^{\bar{\theta}_j} \equiv (\xi_i^{\bar{\theta}_j} : \Theta \to \mathbb{R})_{i \in S}$ such that*

(i) $\sum_{i \in S} \xi_i^{\bar{\theta}_j}(\theta) = 0$ *for all* $\theta \in \Theta$,

(ii) $\sum_{\theta_{-i} \in \Theta_{-i}} \xi_i^{\bar{\theta}_j}(\theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i) = 0$ *for all* $i \in S$ *and* $\theta_i \in \Theta_i$,

(iii) $\sum_{\theta_{-j} \in \Theta_{-j}} \xi_j^{\bar{\theta}_j}(\hat{\theta}_j, \theta_{-j}) p(\theta_{-j}|\bar{\theta}_j) < 0$ *for all* $\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}$.

Proof. With the vectors defined in this section, we construct matrices $B \in \mathbb{R}^{m \times l}$ and $D \in \mathbb{R}^{k \times l}$ respectively, where $m = |\Theta_j| - 1$, $k = \sum_{i \in S} |\Theta_i| + |\Theta|$, and $l = n|\Theta|$. Matrix B is obtained by vertically stacking up m row vectors $p^{\{j\}}_{\bar{\theta}_j \hat{\theta}_j} \in \mathbb{R}^l_+$ for all $\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}$. Construct matrix D by stacking up $\sum_{i \in S} |\Theta_i|$ row vectors $p^{\{i\}}_{\theta_i \theta_i} \in \mathbb{R}^l_+$ for all $i \in S$ and $\theta_i \in \Theta_i$ as well as $|\Theta|$ row vectors $e^S_\theta \in \mathbb{R}^l_+$ for all $\theta \in \Theta$.

Suppose by way of contradiction that there is no transfer rule $\xi^{\bar{\theta}_j}$ satisfying the three conditions stated in the lemma. Notice that within each row of B and D, the dimensions that correspond to an agent out of S are equal to zero. Then we can claim that $Bx < \mathbf{0}_{m \times 1}$, $Dx = \mathbf{0}_{k \times 1}$ has no column vector solution $x \in \mathbb{R}^l$. By Theorem 1, there are column vectors $y_1 \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ and $y_2 \in \mathbb{R}^k$, such that $B'y_1 + D'y_2 = \mathbf{0}_{l \times 1}$, or equivalently $-y_2' D = y_1' B$ where both sides are row vectors in $\mathbb{R}^l$. As a result, there exists a profile of nonnegative numbers $(c_{\hat{\theta}_j} \in \mathbb{R}_+)_{\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}}$ with $c_{\hat{\theta}_j} \in \mathbb{R}_{++}$ for some $\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}$ and two profiles of numbers $(a_{\theta_i} \in \mathbb{R})_{\theta_i \in \Theta_i, i \in S}$ and $(b_\theta \in \mathbb{R})_{\theta \in \Theta}$, such that

$$\sum_{i \in S} \sum_{\theta_i \in \Theta_i} a_{\theta_i} p^{\{i\}}_{\theta_i \theta_i} + \sum_{\theta \in \Theta} b_\theta e^S_\theta = \sum_{\hat{\theta}_j \in \Theta_j \backslash \{\bar{\theta}_j\}} c_{\hat{\theta}_j} p^{\{j\}}_{\bar{\theta}_j \hat{\theta}_j}, \tag{7}$$

where both sides are row vectors in $\mathbb{R}^l$.

Fix any $\hat{\theta}_j$ with $c_{\hat{\theta}_j} \neq 0$, $i \in S$ with $i \neq j$, and $\theta_{-j} \in \Theta_{-j}$.

Recall that on each side of expression (7), each dimension in the row vector corresponds to an agent and a type profile. From the dimensions corresponding to j and $(\bar{\theta}_j, \theta_{-j})$ on both sides of (7), we have $a_{\bar{\theta}_j} p(\bar{\theta}_j, \theta_{-j}) + b_{(\bar{\theta}_j, \theta_{-j})} = 0$; from the dimensions corresponding to i and $(\bar{\theta}_j, \theta_{-j})$, we have $a_{\theta_i} p(\bar{\theta}_j, \theta_{-j}) + b_{(\bar{\theta}_j, \theta_{-j})} = 0$. As p has full support, $a_{\bar{\theta}_j} = a_{\theta_i}$.

Similarly, by focusing on the dimensions corresponding to j and $(\hat{\theta}_j, \theta_{-j})$ and corresponding to i and $(\hat{\theta}_j, \theta_{-j})$ on both sides of expression (7), we have $a_{\hat{\theta}_j} p(\hat{\theta}_j, \theta_{-j}) + b_{(\hat{\theta}_j, \theta_{-j})} = c_{\hat{\theta}_j} p(\bar{\theta}_j, \theta_{-j})$ and $a_{\theta_i} p(\hat{\theta}_j, \theta_{-j}) + b_{(\hat{\theta}_j, \theta_{-j})} = 0$. By the observation from the previous paragraph, we have $(a_{\hat{\theta}_j} - a_{\bar{\theta}_j}) p(\hat{\theta}_j, \theta_{-j}) = c_{\hat{\theta}_j} p(\bar{\theta}_j, \theta_{-j})$.

As $(a_{\hat{\theta}_j} - a_{\bar{\theta}_j}) p(\hat{\theta}_j, \theta_{-j}) = c_{\hat{\theta}_j} p(\bar{\theta}_j, \theta_{-j})$ holds for all $\theta_{-j} \in \Theta_{-j}$, it must be the case that $(a_{\hat{\theta}_j} - a_{\bar{\theta}_j}) p(\hat{\theta}_j) = c_{\hat{\theta}_j} p(\bar{\theta}_j)$. Since $c_{\hat{\theta}_j} \neq 0$, we must have $p(\cdot|\hat{\theta}_j) = p(\cdot|\bar{\theta}_j)$, a contradiction with the supposition of the lemma. □

At last, we review two conditions on the prior introduced by Crémer and McLean (1988) and Kosenok and Severinov (2008) to guarantee the existence of first-best mechanisms.

Definition 2. *The prior p is said to satisfy the **Convex Independence** condition if for all $i \in I$ and $\theta_i \in \Theta_i$, $p(\cdot|\theta_i) \notin \text{con}\{p(\cdot|\hat{\theta}_i) : \hat{\theta}_i \in \Theta_i \backslash \{\theta_i\}\}$.*

Definition 3. *The prior p is said to satisfy the **Identifiability** condition if for any full-support $p' \in \Delta(\Theta)$ with $p' \neq p$, there exists $i \in I$ and $\theta_i \in \Theta_i$, for which $p'(\cdot|\theta_i) \notin \text{con}\{p(\cdot|\hat{\theta}_i) : \hat{\theta}_i \in \Theta_i\}$.*

A.2 Proof of Proposition 1

Proof of Lemma 1. Fix any $p \in \Delta(\Theta)$, $i \in I$, $\bar{\theta} \in \Theta$, and $\epsilon \in (0, \frac{2p(\bar{\theta}_i)(1-p(\bar{\theta}_i))}{2p(\bar{\theta}_i)(1-p(\bar{\theta}_i))+3|\Theta|})$.

Step 1. Construct a payoff structure and an efficient allocation rule q.

The set of feasible outcomes is $A = \Delta\{x_1, x_2\}$. Agents' payoffs from a fixed outcome only depend on the type of agent i and are given in the following table. In each parenthesis, the first component is the payoff of agent i and the second one is that of each $j \in I \backslash \{i\}$. Let $u_0(a) = 0$ for all $a \in A$.

(u_i, u_j)	x_1	x_2
$\theta_i = \bar{\theta}_i$	$(1, 1)$	$(2-\epsilon, \frac{n-2}{n-1})$
$\theta_i \neq \bar{\theta}_i$	$(2-\epsilon, \frac{n-2}{n-1})$	$(1, 1)$

Table 4: Agents' utility functions

Let q be the unique efficient allocation rule: $q(\theta) = x_1$ if $\theta_i = \bar{\theta}_i$, and $q(\theta) = x_2$ elsewhere. The outcome assigned by q changes only when some type-θ_i agent i, where $\theta_i \in \Theta_i \backslash \{\bar{\theta}_i\}$, misreports $\bar{\theta}_i$, or the other way around.

Step 2. Suppose by way of contradiction that the desired inequality does not hold and reach a contradiction.

For each coalition S and type profiles θ_S, θ'_S, we label the constraint

$$V_S[q, t](\theta_S, \theta_S) \geqslant V_S[q, t](\theta_S, \theta'_S)$$

by CIC$(\theta_S; \theta'_S)$, where CIC standards for coalition incentive compatibility.[14]

Suppose by way of contradiction that there exists a feasible mechanism (q, t) satisfying the following three properties.

Property (i), IC of agent i.

In particular, for types $\theta_i \neq \theta'_i$ where either θ_i or θ'_i is equal to $\bar{\theta}_i$, the IC constraints of type θ_i require

$$\text{IC}(\theta_i; \theta'_i): \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i) - \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\theta'_i, \theta_{-i}) p(\theta_{-i}|\theta_i) \geqslant 1 - \epsilon, \tag{8}$$

where the right-hand-side expression is the change in agent i's payoff due to the changed outcome.

Moreover, for types $\theta_i \neq \theta'_i$ where $\theta_i, \theta'_i \in \Theta_i \backslash \{\bar{\theta}_i\}$,

$$\text{IC}(\theta_i; \theta'_i): \qquad \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i) - \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\theta'_i, \theta_{-i}) p(\theta_{-i}|\theta_i) \geqslant 0.$$

[14]This constraint essentially treats coalition S as a "pseudo" agent with type θ_S. This "pseudo" agent's utility is the sum of its members' utility levels. The CIC$(\theta_S; \theta'_S)$ constraint requires that type-θ_S "pseudo" agent S does not benefit from misreporting θ'_S.

Property (ii), CIC of coalition $I\backslash\{i\}$ (or IC if $n=2$).

For each pair of $\theta_{-i} \neq \theta'_{-i}$,

$$\text{CIC}(\theta_{-i};\theta'_{-i}): \qquad \sum_{j\in I\backslash\{i\}}\sum_{\theta_i\in\Theta_i} t_j(\theta_i,\theta_{-i})p(\theta_i|\theta_{-i}) - \sum_{j\in I\backslash\{i\}}\sum_{\theta_i\in\Theta_i} t_j(\theta_i,\theta'_{-i})p(\theta_i|\theta_{-i}) \geqslant 0.$$

Property (iii), CIC of coalition I.

For each type profile $\theta=(\bar\theta_i,\theta_{-i})\in\Theta\backslash\{\bar\theta\}$,

$$\text{CIC}(\bar\theta;\theta): \qquad \sum_{j\in I} t_j(\bar\theta) - \sum_{j\in I} t_j(\theta) \geqslant 0;$$

For each type profile $\theta=(\theta_i,\theta_{-i})\in\Theta\backslash\{\bar\theta\}$ where $\theta_i\neq\bar\theta_i$,

$$\text{CIC}(\bar\theta;\theta): \qquad \sum_{j\in I} t_j(\bar\theta) - \sum_{j\in I} t_j(\theta) \geqslant 2-\epsilon+(n-1)\frac{n-2}{n-1}-n = -\epsilon.$$

For each type profile $\theta=(\bar\theta_i,\theta_{-i})\in\Theta\backslash\{\bar\theta\}$,

$$\text{CIC}(\theta;\bar\theta): \qquad \sum_{j\in I} t_j(\theta) - \sum_{j\in I} t_j(\bar\theta) \geqslant 0.$$

For each type profile $\theta=(\theta_i,\theta_{-i})\in\Theta\backslash\{\bar\theta\}$ where $\theta_i\neq\bar\theta_i$,

$$\text{CIC}(\theta;\bar\theta): \qquad \sum_{j\in I} t_j(\theta) - \sum_{j\in I} t_j(\bar\theta) \geqslant 2-\epsilon+(n-1)\frac{n-2}{n-1}-n = -\epsilon.$$

Now we scale each constraint $\text{IC}(\theta_i;\theta'_i)$ where $\theta_i\neq\theta'_i$ by the factor $p(\theta_i)p(\theta'_i)$. For example, after scaling, expression (8) becomes:

$$\sum_{\theta_{-i}\in\Theta_{-i}} t_i(\theta_i,\theta_{-i})p(\theta_i,\theta_{-i})p(\theta'_i) - \sum_{\theta_{-i}\in\Theta_{-i}} t_i(\theta'_i,\theta_{-i})p(\theta_i,\theta_{-i})p(\theta'_i) \geqslant (1-\epsilon)p(\theta_i)p(\theta'_i).$$

Similarly, we scale each constraint $\text{CIC}(\theta_{-i};\theta'_{-i})$ where $\theta_{-i}\neq\theta'_{-i}$ by $p(\theta_{-i})p(\theta'_{-i})$, each $\text{CIC}(\bar\theta;\theta)$ where $\theta\in\Theta\backslash\{\bar\theta\}$ by $|p(\theta)-p(\theta_i)p(\theta_{-i})|+p(\theta)-p(\theta_i)p(\theta_{-i})$, and each $\text{CIC}(\theta;\bar\theta)$ where $\theta\in\Theta\backslash\{\bar\theta\}$ by $|p(\theta)-p(\theta_i)p(\theta_{-i})|$.

Then aggregate these scaled constraints. We can cancel all terms containing t on the left-hand side. We eventually have $0\geqslant 2(1-\epsilon)p(\bar\theta_i)(1-p(\bar\theta_i))-\epsilon\sum_{\theta_{-i}\in\Theta_{-i}}\sum_{\theta_i\in\Theta_i\backslash\{\bar\theta_i\}}[2|p(\theta)-p(\theta_i)p(\theta_{-i})|+p(\theta)-p(\theta_i)p(\theta_{-i})] > 2(1-\epsilon)p(\bar\theta_i)(1-p(\bar\theta_i))-3\epsilon|\Theta|>0$, where the first strict inequality uses the observation that

$$\begin{aligned}&\sum_{\theta_{-i}\in\Theta_{-i}}\sum_{\theta_i\in\Theta_i\backslash\{\bar\theta_i\}}[2|p(\theta)-p(\theta_i)p(\theta_{-i})|+p(\theta)-p(\theta_i)p(\theta_{-i})]\\ \leqslant&\sum_{\theta_{-i}\in\Theta_{-i}}\sum_{\theta_i\in\Theta_i\backslash\{\bar\theta_i\}}[3\underbrace{|p(\theta)-p(\theta_i)p(\theta_{-i})|}_{\leqslant 1}]\leqslant 3(|\Theta_i|-1)||\Theta_{-i}|<3|\Theta|\end{aligned}$$

and the second strict inequality uses the range of ϵ specified at the beginning of the proof. This leads to a contradiction. $\square$

Proof of Proposition 1. Fix an information structure (Θ, p). Lemma 1 constructs a payoff structure and establishes that for any standard Bayesian mechanism (q, t) that achieves FSE, there exists $S \in \{I \backslash \{n\}, I\}$ and $\theta_S \neq \theta'_S \in \Theta_S$ such that $V_S[q,t](\theta_S, \theta'_S) > V_S[q,t](\theta_S, \theta_S)$. We now fix one mechanism (q, t) that achieves FSE and one such S for the remainder of this proof.

Among all $\delta^S : \Theta_S \to \Delta(\Theta_S)$, let $\hat{\delta}^S$ maximize $V_S[q,t](\tilde{\theta}_S, \delta^S)$ for each $\tilde{\theta}_S \in \Theta_S$. Hence, following $\hat{\delta}^S$ is ex-ante more profitable for S than being truthful, i.e.,

$$\sum_{\theta_S \in \Theta_S} V_S[q,t](\theta_S, \hat{\delta}^S) p(\theta_S) > \sum_{\theta_S \in \Theta_S} V_S[q,t](\theta_S, \bar{\delta}^S) p(\theta_S). \tag{9}$$

Let $(\Theta_S, \hat{p})$ be the information structure in the sub-environment with agents in S only, where $\hat{p}$ is the marginal distribution of p over Θ_S.

Step 1. Suppose $\hat{p}$ satisfies the Convex Independence condition and the Identifiability condition (defined in Appendix A.1). Now we construct a feasible S-communicative manipulation.

Let $\mathbb{T}$ be a compact subset of $\mathbb{R}$ such that $t_i(\theta) \in \mathbb{T}$ for all $i \in I$ and $\theta \in \Theta$. For each $i \in S$, define a utility function $\hat{u}_i$ over type space Θ_S and an extended set of feasible outcomes $\tilde{A} \equiv \Delta(A \times \mathbb{T}^n)$ as follows: for each $\theta_S \in \Theta_S$ and pair $(\bar{a}, \bar{t}) \in A \times \mathbb{T}^n$, $\hat{u}_i((\bar{a}, \bar{t}), \theta_S) \equiv \sum_{\theta_{-S} \in \Theta_{-S}} [u_i(\bar{a}, (\theta_S, \theta_{-S})) + \bar{t}_i] p(\theta_{-S}|\theta_S)$, and then extend the definition of $\hat{u}_i$ to $\Delta(A \times \mathbb{T}^n)$ by the standard expected utility.

Define an allocation rule $\hat{q} : \Theta_S \to \tilde{A}$ such that $\hat{q}(\theta_S)$ is a lottery that is equal to $(q(\theta'_S, \theta_{-S}), t(\theta'_S, \theta_{-S}))$ with probability $\hat{\delta}^S[\theta_S](\theta'_S) p(\theta_{-S}|\theta_S)$ for each $\theta_S, \theta'_S \in \Theta_S$ and $\theta_{-S} \in \theta_{-S}$. FSE requires that the right-hand side of expression (9) is equal to 0. Hence, (9) implies that $\sum_{i \in S} \sum_{\theta_S \in \Theta_S} \hat{u}_i(\hat{q}(\theta_S), \theta_S) \hat{p}(\theta_S) = \sum_{\theta_S \in \Theta_S} V_S[q,t](\theta_S, \hat{\delta}^S) p(\theta_S) > 0$. Then by Kosenok and Severinov (2008), in the sub-environment, there exists an ex-post budget balanced transfer rule, $\psi^S : \Theta_S \to \mathbb{R}^{|S|}$, such that $(\hat{q}, \psi^S)$ is interim IR and IC. Notice that for each $i \in S$, θ_i, $\hat{\theta}_i \in \Theta_i$,

$$\sum_{\theta_{S \backslash \{i\}} \in \Theta_{S \backslash \{i\}}} [\hat{u}_i(\hat{q}(\hat{\theta}_i, \theta_{S \backslash \{i\}}), (\theta_i, \theta_{S \backslash \{i\}})) + \psi_i^S(\hat{\theta}_i, \theta_{S \backslash \{i\}})] \hat{p}(\theta_{S \backslash \{i\}}|\theta_i) = V_i[q^{\hat{\delta}_S}, t^{\hat{\delta}_S} + \psi^S](\theta_i, \hat{\theta}_i).$$

The interim IR and IC of $(\hat{q}, \psi^S)$ in the sub-environment imply that the S-communicative manipulation $(q^{\hat{\delta}_S}, t^{\hat{\delta}_S} + \psi^S)$ in the original environment satisfies S-IR and S-IC, and thus, is S-feasible.

Step 2. Show that (q, t) does not satisfy RCP.

Since q is efficient, (9) implies that the S-communicative manipulation either decreases the ex-ante payoff of the MD, or hurts at least one agent out of S. In the latter case, the S-communicative manipulation leads to an infeasible mechanism. In either case, (q, t) does not satisfy RCP with respect to S, and thus not RCP.

Step 3. Establish the conclusion.

Recall that we imposed that $\hat{p}$ satisfies the Convex Independence condition and the Identifiability condition. In the environment $(\Theta_S, \hat{p})$, when $|\Theta_i| \leqslant |\Theta_{S\backslash\{i\}}|$ for all $i \in S$, almost every prior $\hat{p} \in \Delta(\Theta)$ satisfies the Convex Independence condition. By Kosenok and Severinov (2008), when $|S| = 3$ and $|\Theta_S| \geqslant 12$ (i.e., $|S| > 3$ or $|S| = 3$ but there exists an agent with at least three types), almost every prior $\hat{p} \in \Delta(\Theta_S)$ satisfies the Identifiability condition. Notice that $S \in \{I\backslash\{n\}, I\}$. Hence, under the dimensional restrictions on the type space stated in Proposition 1, for almost all $p \in \Delta(\Theta)$, the $\hat{p}$ satisfies these conditions in the sub-environment, irrespective of whether $S = I\backslash\{n\}$ or $S = I$. □

A.3 Proof of Proposition 2

Lemma 5. *Suppose the BDP property holds for the prior p. Let (q, T) be an ambiguous mechanism that extracts the full surplus. If there exists $S \in 2^I\backslash\{\varnothing\}$ with $|S| \geqslant 2$ and δ^S such that $\sum_{\theta_S \in \Theta_S} V_S[q, t](\theta_S, \delta^S)p(\theta_S) > 0$ for all $t \in T$, then (q, T) does not satisfy RCP*.*

Proof. **Step 1.** For each $t \in T$, show that there exists $\zeta^t \equiv (\zeta_i^t : \Theta_S \to \mathbb{R})_{i \in S}$ such that

(a) $\sum_{i \in S} \zeta_i^t(\theta_S) = 0$ for all $\theta_S \in \Theta_S$;

(b) $\sum_{\theta_{S\backslash\{i\}} \in \Theta_{S\backslash\{i\}}} \zeta_i^t(\theta_i, \theta_{S\backslash\{i\}}) p(\theta_{S\backslash\{i\}}|\theta_i) = w_i(t, \theta_i)$ for all $i \in S$ and $\theta_i \in \Theta_i$, where

$$w_i(t, \theta_i) \equiv -V_i[q^{\delta^S}, t^{\delta^S}](\theta_i, \theta_i) + \frac{1}{|S|} \sum_{\tilde{\theta}_S \in \Theta_S} V_S[q, t](\tilde{\theta}_S, \delta^S) p(\tilde{\theta}_S). \tag{10}$$

As $\sum_{i\in S}\sum_{\theta_i\in\Theta_i} w_i(t,\theta_i)p(\theta_i)=0$, we apply Lemma A3 of Kosenok and Severinov (2008) in a sub-environment with agents in S and type space Θ_S to establish the existence of ζ^t.

Step 2. Construct an ambiguous S-side contract (δ^S,Ψ^S).

Since the BDP property holds, for each $j\in S$ and $\bar{\theta}_j\in\Theta_j$, there exists a transfer rule $\xi^{\bar{\theta}_j}\equiv(\xi_i^{\bar{\theta}_j}:\Theta\to\mathbb{R})_{i\in S}$, satisfying the conditions in Lemma 4. Fix any $\lambda\in\mathbb{R}_+$ that is weakly larger than

$$\max_{\substack{t\in T,j\in S,\bar{\theta}_j\in\Theta_j,\\ \hat{\theta}_j\in\Theta_j\setminus\{\bar{\theta}_j\}}}\frac{V_j[q^{\delta^S},t^{\delta^S}](\bar{\theta}_j,\bar{\sigma}_j)-V_j[q^{\delta^S},t^{\delta^S}](\bar{\theta}_j,\hat{\theta}_j)+w_j(t,\bar{\theta}_j)-\sum_{\theta_{S\setminus\{j\}}\in\Theta_{S\setminus\{j\}}}\zeta_j^t(\hat{\theta}_j,\theta_{S\setminus\{j\}})p(\theta_{S\setminus\{j\}}|\bar{\theta}_j)}{\sum_{\theta_{-j}\in\Theta_{-j}}\xi_j^{\bar{\theta}_j}(\hat{\theta}_j,\theta_{-j})p(\theta_{-j}|\bar{\theta}_j)}.$$

For each $j\in S$ and $\bar{\theta}_j\in\Theta_i$, define $\psi^{\bar{\theta}_j}\equiv(\psi_i^{\bar{\theta}_j}:T\times\Theta\to\mathbb{R})_{i\in S}$ by $\psi_i^{\bar{\theta}_j}(t,\theta)\equiv\zeta_i^t(\theta_S)+\lambda\xi_i^{\bar{\theta}_j}(\theta)$ for all $t\in T$, $\theta\in\Theta$, and $i\in S$. Define $\Psi^S\equiv\{\psi^{\bar{\theta}_j}|j\in S,\bar{\theta}_j\in\Theta_j\}$, where each transfer rule is ex-post budget balanced within S due to (a) from Step 1 and (i) in Lemma 4.

Step 3. Verify the S-feasibility of S-communicative manipulation $(q^{\delta^S},T^{\delta^S}+\Psi^S)$.

Fix any $i\in S$ and $\theta_i\in\Theta_i$ throughout this step.

By (10) and the supposition of this lemma, for each $t\in T$,

$$V_i[q^{\delta^S},t^{\delta^S}](\theta_i,\theta_i)+w_i(t,\theta_i)=\frac{1}{|S|}\sum_{\tilde{\theta}_S\in\Theta_S}V_S[q,t](\tilde{\theta}_S,\delta^S)p(\tilde{\theta}_S)>0. \tag{11}$$

Notice that for each $\psi^S\in\Psi^S$, there exists $j\in S$ and $\bar{\theta}_j\in\Theta_j$ such that $\psi_i^S(t,(\theta_i,\theta_{-i}))=\zeta_i^t(\theta_i,\theta_{S\setminus\{i\}})+\lambda\xi_i^{\bar{\theta}_j}(\theta_i,\theta_{-i})$ for all $\theta_{-i}\in\Theta_{-i}$ and $t\in T$. For convenience, for each $t\in T$, we denote $V_i[q^{\delta^S},t^{\delta^S}+\psi^S(t,\cdot)](\theta_i,\theta_i)\equiv V_i[q^{\delta^S},t^{\delta^S}](\theta_i,\theta_i)+\sum_{\theta_{-i}\in\Theta_{-i}}\psi_i^S(t,\theta_i,\theta_{-i})p(\theta_{-i}|\theta_i)$. Hence, for $\psi^S\in\Psi^S$ such that $\psi^S(t,\theta)=\zeta^t(\theta_S)+\lambda\xi^{\bar{\theta}_j}(\theta)$ for all $t\in T$ and $\theta\in\Theta$,

$$\begin{aligned}&\min_{t\in T}\{V_i[q^{\delta^S},t^{\delta^S}+\psi^S(t,\cdot)](\theta_i,\theta_i)\}\\ =&\min_{t\in T}\{V_i[q^{\delta^S},t^{\delta^S}](\theta_i,\theta_i)+\sum_{\theta_{-i}\in\Theta_{-i}}[\zeta_i^t(\theta_i,\theta_{S\setminus\{i\}})+\lambda\xi_i^{\bar{\theta}_j}(\theta_i,\theta_{-i})]p(\theta_{-i}|\theta_i)\}\\ =&\min_{t\in T}\{\underbrace{V_i[q^{\delta^S},t^{\delta^S}](\theta_i,\theta_i)+w_i(t,\theta_i)}_{\text{independent of }\psi^S\in\Psi^S\text{ and positive by (11)}}\}=V_i[q^{\delta^S},T^{\delta^S}+\Psi^S](\theta_i,\bar{\sigma}_i)>0.\end{aligned} \tag{12}$$

Notice that the second equality follows from Condition (b) of ζ^t established from Step 1 and Condition (ii) of $\xi^{\bar{\theta}_j}$ stated in Lemma 4. We have established the S-IR constraint for θ_i.

On the other hand, for each $\sigma_i : \Theta_i \to \Delta(\Theta_i)$ such that $\sigma_i \neq \bar{\sigma}_i$, since $\psi^{\theta_i} \in \Psi^S$,

$$
\begin{aligned}
&V_i[q^{\delta^S}, T^{\delta^S} + \Psi^S](\theta_i, \sigma_i) \\
\leqslant &\min_{t \in T}\{ \sum_{\hat{\theta}_i \in \Theta_i} [V_i[q^{\delta^S}, t^{\delta^S}](\theta_i, \hat{\theta}_i) + \sum_{\theta_{-i} \in \Theta_{-i}} [\zeta_i^t(\hat{\theta}_i, \theta_{S\setminus\{i\}}) + \lambda \xi_i^{\theta_i}(\hat{\theta}_i, \theta_{-i})] p(\theta_{-i}|\theta_i)]\sigma_i[\theta_i](\hat{\theta}_i)\} \\
\leqslant &\min_{t \in T}\{V_i[q^{\delta^S}, t^{\delta^S}](\theta_i, \theta_i) + w_i(t, \theta_i)\} \overset{(12)}{=} V_i[q^{\delta^S}, T^{\delta^S} + \Psi^S](\theta_i, \bar{\sigma}_i),
\end{aligned}
$$

where the second inequality follows from the choice of λ. To this end, we have established the S-IC constraint for θ_i.

Similarly, we can verify the S-IR and S-IC constraints for each member in S and each type. Thus, $(q^{\delta^S}, T^{\delta^S} + \Psi^S)$ is an S-feasible ambiguous S-communicative manipulation.

Step 4. Show that (q, T) does not satisfy RCP*.

Fix any $t \in T$, since $\sum_{\theta_S \in \Theta_S} V_S[q, t](\theta_S, \delta^S) p(\theta_S) > 0$, the efficiency of q implies that in this ambiguous S-communicative manipulation, either the MD does not extract the full surplus or the IR constraint of an agent out of S is violated. Hence, (q, T) does not satisfy RCP*. □

Lemma 6. *If the CBDP property fails, then there exists a payoff structure under which there does not exist an FSE ambiguous mechanism that satisfies the RCP* condition.*

Proof. **Step 1.** Discuss the case when the BDP property doesn't not hold for the prior p.

Suppose there exists $i \in I$ and types $\bar{\theta}_i \neq \hat{\theta}_i$ such that $p(\cdot|\bar{\theta}_i) = p(\cdot|\hat{\theta}_i)$. Consider the payoff structure in the proof of Proposition 1 with the modification that $\epsilon \in (0, 1)$ and the same efficient allocation rule q. If there exists an FSE ambiguous mechanism (q, T), then the following two IC constraints must hold:

$$
\begin{aligned}
\mathrm{IC}(\bar{\theta}_i; \hat{\theta}_i): \quad &\min_{t \in T}\{1 + \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\bar{\theta}_i, \theta_{-i}) p(\theta_{-i}|\bar{\theta}_i)\} \geqslant \min_{t \in T}\{2 - \epsilon + \sum_{\theta_{-S} \in \Theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i}) p(\theta_{-i}|\bar{\theta}_i)\}, \\
\mathrm{IC}(\hat{\theta}_i; \bar{\theta}_i): \quad &\min_{t \in T}\{1 + \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i}) p(\theta_{-i}|\hat{\theta}_i)\} \geqslant \min_{t \in T}\{2 - \epsilon + \sum_{\theta_{-i} \in \Theta_{-i}} t_i(\bar{\theta}_i, \theta_{-i}) p(\theta_{-i}|\hat{\theta}_i)\}.
\end{aligned}
$$

In these inequalities, the terms independent of t can be moved out of the minimization operators.

We can sum up the two IC constraints, cancel all terms containing t, and obtain $2 \geqslant 4 - 2\epsilon$. This contradicts the restriction that $\epsilon \in (0, 1)$.

Step 2. Discuss the case when the BDP property holds but the CBDP property doesn't not hold for prior p.

Suppose for some non-singleton $S \in 2^I \backslash \{\varnothing, I\}$, there are two type profiles $\theta_S^1 \neq \theta_S^2$ with $p(\theta_S^1) \leqslant p(\theta_S^2)$ such that $p(\cdot|\theta_S^1) = p(\cdot|\theta_S^2)$. Denote an agent in S whose types are different under θ_S^1 and θ_S^2 by i, and label his component in θ_S^1 by $\bar{\theta}_i$. Consider the payoff structure in the proof of Proposition 1 except that $\epsilon \in (0, \frac{1}{n-1})$.

Let δ^S be the truthful joint reporting strategy except that (i) $\delta^S[\theta_S^1](\theta_S^2) = 1$, (ii) $\delta^S[\theta_S^2](\theta_S^1) = p(\theta_S^1)/p(\theta_S^2)$ and $\delta^S[\theta_S^2](\theta_S^2) = 1 - p(\theta_S^1)/p(\theta_S^2)$. It is easy to see that

$$p(\theta) = \sum_{\bar{\theta}_S \in \Theta_S} p(\bar{\theta}_S, \theta_{-S})\delta^S[\bar{\theta}_S](\theta_S), \forall \theta \in \Theta.$$

For any FSE ambiguous mechanism (q, T), it must be the case that

$$\sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} \sum_{i \in S} t_i(\theta'_S, \theta_{-S}) p(\theta'_S, \theta_{-S}) = -|S|, \ \forall t \in T.$$

Hence, by the observation established from the previous paragraph,

$$\sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} \sum_{i \in S} t_i(\theta'_S, \theta_{-S}) \sum_{\theta_S \in \Theta_S} p(\theta_S, \theta_{-S})\delta^S[\theta_S](\theta'_S) = -|S|, \forall t \in T. \tag{13}$$

Notice that for each $t \in T$, $\sum_{\theta_S \in \Theta_S} V_S[q, t](\theta_S, \delta^S)p(\theta_S)$ can be equivalently written as

$$\sum_{\theta_S \in \Theta_S} \sum_{i \in S} \sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} [u_i\big(q(\theta'_S, \theta_{-S}), (\theta_S, \theta_{-S})\big) + t_i(\theta'_S, \theta_{-S})] p(\theta_S, \theta_{-S})\delta^S[\theta_S](\theta'_S)$$
$$\overset{(13)}{=} \sum_{\theta_S \in \Theta_S} \sum_{i \in S} \sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} u_i\big(q(\theta'_S, \theta_{-S}), (\theta_S, \theta_{-S})\big) p(\theta_S, \theta_{-S})\delta^S[\theta_S](\theta'_S) - |S| > 0.$$

The inequality holds, because given the payoff structure constructed in Step 2, the range of ϵ, the fact that $|S| < n$, and the definition of q, for all $\theta_S, \theta'_S \in \Theta_S$ and $\theta_{-S} \in \Theta_{-S}$,

$$\sum_{i \in S} u_i\big(q(\theta'_S, \theta_{-S}), (\theta_S, \theta_{-S})\big) \geqslant |S|$$

and the strict inequality holds for some θ_S, θ'_S, and θ_{-S} such that $p(\theta_S, \theta_{-S})\delta^S[\theta_S](\theta'_S) > 0$ (since there is a misreport between type $\bar{\theta}_i$ and another type in $\Theta_i \backslash \{\bar{\theta}_i\}$). As a result, $\sum_{\theta_S \in \Theta_S} V_S[q, t](\theta_S, \delta^S)p(\theta_S) > 0$ for all $t \in T$. By Lemma 5, (q, T) does not satisfy RCP*.

Hence, there does not exist any FSE ambiguous mechanism satisfying RCP*. □

Proof of Lemma 3. The truthful joint reporting strategy is $\bar{\delta}^S : \Theta_S \to \Delta(\Theta_S)$. Denote all non-truthful deterministic joint reporting strategies by δ^{S1}, δ^{S2},..., $\delta^{S\bar{k}}$, where $\bar{k}$ is a positive integer. For each $\delta^S : \Theta_S \to \Delta(\Theta_S)$, denote $p[\delta^S] \equiv \sum_{\bar{\theta}_S, \hat{\theta}_S \in \Theta_S} \delta^S[\bar{\theta}_S](\hat{\theta}_S) p^S_{\bar{\theta}_S \hat{\theta}_S} \in \mathbb{R}^{n|\Theta|}_+$.

Step 1. Show that there do not exist weights β_1, ..., $\beta_{\bar{k}} \geqslant 0$ such that $\beta_1 + ... + \beta_{\bar{k}} = 1$, for which

$$p[\bar{\delta}^S] = \beta_1 p[\delta^{S1}] + ... + \beta_{\bar{k}} p[\delta^{S\bar{k}}]. \tag{14}$$

Suppose by way of contradiction that there exist weights $\beta_1, ..., \beta_{\bar{k}}$ such that the above equation holds. We define $\tilde{\delta}^S \equiv \beta_1 \delta^{S1} + ... + \beta_{\bar{k}} \delta^{S\bar{k}}$. By linearity of $p[\delta^S]$ in δ^S, definition of $\tilde{\delta}^S$, and expression (14), it must be true that $p[\bar{\delta}^S] = p[\tilde{\delta}^S]$. According to the definition of the two vectors, we have $\sum_{\theta_S \in \Theta_S} p^S_{\theta_S \theta_S} = \sum_{\bar{\theta}_S, \theta_S \in \Theta_S} \tilde{\delta}^S[\bar{\theta}_S](\theta_S) p^S_{\bar{\theta}_S \theta_S} \in \mathbb{R}^{n|\Theta|}_+$. As a result,

$$p(\theta_S)p(\theta_{-S}|\theta_S) = \sum_{\bar{\theta}_S \in \Theta_S} \tilde{\delta}^S[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S) p(\theta_{-S}|\bar{\theta}_S), \forall \theta_S \in \Theta_S, \theta_{-S} \in \Theta_{-S},$$

This implies that

$$p(\cdot|\theta_S) = \sum_{\bar{\theta}_S \in \Theta_S} \underbrace{\Big[\frac{p(\bar{\theta}_S)}{p(\theta_S)} \tilde{\delta}^S[\bar{\theta}_S](\theta_S)\Big]}_{\geqslant 0} p(\cdot|\bar{\theta}_S), \forall \theta_S \in \Theta_S. \tag{15}$$

Since the S-BDP property holds, there is a one-to-one correspondence between elements of Θ_S and $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta_S\}$. Let $\Theta^1_S \subseteq \Theta_S$ be the set such that $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta^1_S\}$ is the set of all extreme points of $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta_S\}$. Recursively, for $m = 2, ..., \bar{m}$, let $\Theta^m_S \subseteq \Theta_S \backslash (\Theta^1_S \cup ... \cup \Theta^{m-1}_S)$ be the set such that $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta^m_S\}$ is the set of all extreme points of $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta_S \backslash (\Theta^1_S \cup ... \cup \Theta^{m-1}_S)\}$. Since Θ_S is finite, it takes $\bar{m} < \infty$ rounds, such that $\Theta^1_S \cup ... \cup \Theta^{\bar{m}}_S = \Theta_S$. Then $\{\Theta^1_S, \Theta^2_S, ..., \Theta^{\bar{m}}_S\}$ is a finite partition of Θ_S.

We claim that $\tilde{\delta}^S[\theta_S](\theta_S) = 1$ for all $\theta_S \in \Theta^1_S$. To see this, fix any $\theta_S \in \Theta^1_S$. Since $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta^1_S\}$ is the set of all extreme points of $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta_S\}$, by expression (15), $\frac{p(\bar{\theta}_S)}{p(\theta_S)} \tilde{\delta}^S[\bar{\theta}_S](\theta_S) = 0$, and thus, $\tilde{\delta}^S[\bar{\theta}_S](\theta_S) = 0$, for any $\bar{\theta}_S \in \Theta_S \backslash \{\theta_S\}$, and $\frac{p(\theta_S)}{p(\theta_S)} \tilde{\delta}^S[\theta_S](\theta_S) = \tilde{\delta}^S[\theta_S](\theta_S) = 1$.

We further claim that $\tilde{\delta}^S[\theta_S](\theta_S) = 1$ for all $\theta_S \in \Theta^2_S$. To see this, fix any $\theta_S \in \Theta^2_S$. We have shown in the previous claim that $\delta^S[\bar{\theta}_S](\theta_S) = 0$ for all $\bar{\theta}_S \in \Theta^1_S$. Furthermore, since $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta^2_S\}$ is the set of all extreme points of $\{p(\cdot|\tilde{\theta}_S)|\tilde{\theta}_S \in \Theta_S \backslash \Theta^1_S\}$, by expression (15), $\frac{p(\bar{\theta}_S)}{p(\theta_S)} \tilde{\delta}^S[\bar{\theta}_S](\theta_S) = 0$, and thus, $\tilde{\delta}^S[\bar{\theta}_S](\theta_S) = 0$, for any $\bar{\theta}_S \in (\Theta_S \backslash \Theta^1_S) \backslash \{\theta_S\}$, and $\frac{p(\theta_S)}{p(\theta_S)} \tilde{\delta}^S[\theta_S](\theta_S) = \tilde{\delta}^S[\theta_S](\theta_S) = 1$.

Recursively, we can show that $\tilde{\delta}^S[\theta_S](\theta_S) = 1$ for all $\theta_S \in \Theta_S$. By the definition of $\tilde{\delta}^S$, this further implies that for every $k \in \{1, ..., \bar{k}\}$ with $\beta_k > 0$, δ^{Sk} imposes probability 1 on truthful revealing, a contradiction with the supposition that all δ^{Sk} are non-truthful deterministic joint reporting strategies.

Step 2. Prove that there exists an ex-post budget balanced transfer rule $\phi^S : \Theta \to \mathbb{R}^n$ such that

(i) $\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i^S(\theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i) = 0$ for all $i \in I$ and $\theta_i \in \Theta_i$;

(ii) $\sum_{\hat{\theta}_S \in \Theta_S} \sum_{\bar{\theta}_S \in \Theta_S} \sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i^S(\hat{\theta}_S, \theta_{-S}) p(\bar{\theta}_S, \theta_{-S}) \delta^{Sk}[\bar{\theta}_S](\hat{\theta}_S) < 0$ for $k = 1, ..., \bar{k}$.

Suppose by way of contradiction that there does not exist an ex-post budget balanced transfer rule ϕ^S satisfying Conditions (i) and (ii) above.

With the vectors defined in Appendix A.1, we construct matrices $B \in \mathbb{R}^{\bar{k} \times (n|\Theta|)}$ and $D \in \mathbb{R}^{(\sum_{i \in I} |\Theta_i| + |\Theta|) \times (n|\Theta|)}$ respectively. Matrix B is obtained by vertically stacking up $\bar{k}$ row vectors $\sum_{\bar{\theta}_S, \hat{\theta}_S \in \Theta_S} \delta^{Sk}[\bar{\theta}_S](\hat{\theta}_S) p^S_{\bar{\theta}_S \hat{\theta}_S} \in \mathbb{R}_+^{n|\Theta|}$ for all $k = 1, ..., \bar{k}$. Construct matrix D by stacking up $\sum_{i \in I} |\Theta_i|$ row vectors $p^{\{i\}}_{\theta_i \theta_i} \in \mathbb{R}_+^{n|\Theta|}$ for all $i \in I$ and $\theta_i \in \Theta_i$ as well as $|\Theta|$ row vectors $e^I_\theta \in \mathbb{R}_+^{n|\Theta|}$ for all $\theta \in \Theta$.

Suppose by way of contradiction that there is no transfer rule ϕ^S satisfying requirements stated in Step 2. Then we can claim that $Bx < \mathbf{0}_{\bar{k} \times 1}$, $Dx = \mathbf{0}_{(\sum_{i \in I} |\Theta_i| + |\Theta|) \times 1}$ has no column vector solution $x \in \mathbb{R}^{n|\Theta|}$. By Theorem 1, there are column vectors $y_1 \in \mathbb{R}_+^{\bar{k}} \backslash \{\mathbf{0}\}$ and $y_2 \in \mathbb{R}^{\sum_{i \in I} |\Theta_i| + |\Theta|}$, such that $B' y_1 + D' y_2 = \mathbf{0}_{(n|\Theta|) \times 1}$, or equivalently $-y_2' D = y_1' B$ where both sides are row vectors in $\mathbb{R}^{n|\Theta|}$. As a result, there exists a profile of numbers $(a_{\theta_i} \in \mathbb{R})_{\theta_i \in \Theta_i, i \in I}$, a profile of numbers $(b_\theta \in \mathbb{R})_{\theta \in \Theta}$, and a profile of nonnegative numbers $(c_k \in \mathbb{R}_+)_{k=1,...,\bar{k}}$ with at least one $c_k > 0$, such that

$$\sum_{i \in I} \sum_{\theta_i \in \Theta_i} a_{\theta_i} p^{\{i\}}_{\theta_i \theta_i} + \sum_{\theta \in \Theta} b_\theta e^I_\theta = \sum_{\bar{\theta}_S, \hat{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\hat{\theta}_S) p^S_{\bar{\theta}_S \hat{\theta}_S}. \tag{16}$$

By definitions of vectors in Appendix A.1, $\sum_{\theta \in \Theta} p(\theta) e^I_\theta = \sum_{i \in I} \sum_{\theta_i \in \Theta_i} p^{\{i\}}_{\theta_i \theta_i}$. Multiply this equation by a large $\lambda \in \mathbb{R}_+$, such that $\tilde{a}_{\theta_i} \equiv \lambda - a_{\theta_i} \in \mathbb{R}_{++}$ for all $i \in I$ and $\theta_i \in \Theta_i$ and $\tilde{b}_\theta \equiv \lambda p(\theta) + b_\theta \in \mathbb{R}_+$ for all $\theta \in \Theta$, and add the scaled equation with (16). We have

$$\sum_{\theta \in \Theta} \tilde{b}_\theta e^I_\theta = \sum_{i \in I} \sum_{\theta_i \in \Theta_i} \tilde{a}_{\theta_i} p^{\{i\}}_{\theta_i \theta_i} + \sum_{\bar{\theta}_S, \hat{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\hat{\theta}_S) p^S_{\bar{\theta}_S \hat{\theta}_S}. \tag{17}$$

Recall that $|S| \geqslant 2$. We claim that

$$\tilde{a}_{\theta_i} = \tilde{a}_{\theta_j}, \, \forall i, j \in S \, \text{with} \, i \neq j, \theta_i \in \Theta_i, \, \text{and} \, \theta_j \in \Theta_j. \tag{18}$$

To see this, fix any $\theta = (\theta_S, \theta_{-S}) \in \Theta$, and $i, j \in S$ with $i \neq j$ for now. Each side of (17) is a row vector in $\mathbb{R}^{n|\Theta|}$, and each dimension corresponds to an agent and a type profile. On each side of (17), by focusing on the dimensions corresponding to (i, θ) and (j, θ), we have

$$\begin{aligned}
\tilde{b}_\theta =& \tilde{a}_{\theta_i} p(\theta) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,\ldots,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}) \\
=& \tilde{a}_{\theta_j} p(\theta) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,\ldots,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}).
\end{aligned}$$

The full support assumption of p implies that $\tilde{a}_{\theta_i} = \tilde{a}_{\theta_j}$.

We further claim that

$$\tilde{a}_{\theta_i} = \tilde{a}_{\hat{\theta}_i}, \, \forall i \in S \, \text{and} \, \theta_i, \hat{\theta}_i \in \Theta_i. \tag{19}$$

To see this, fix any two different agents $i, j \in S$, two types of one agent $\theta_i \neq \hat{\theta}_i$, and a type of the other agent $\theta_j \in \Theta_j$ for now. Expression (18) implies that $\tilde{a}_{\theta_i} = \tilde{a}_{\theta_j}$ and $\tilde{a}_{\hat{\theta}_i} = \tilde{a}_{\theta_j}$. As a result, $\tilde{a}_{\theta_i} = \tilde{a}_{\hat{\theta}_i}$.

Expressions (18) and (19) jointly imply that there exists $\kappa \in \mathbb{R}_{++}$ such that $\kappa = \tilde{a}_{\theta_i}$ for all $i \in S$ and $\theta_i \in \Theta_i$.

Fix any $j \notin S$, we further claim that there exists $\kappa' \in \mathbb{R}_{++}$ such that $\kappa' = \tilde{a}_{\theta_j}$ for all $\theta_j \in \Theta_j$. To see this, we assume by way of contradiction that there exist two types $\theta_j \neq \theta'_j$ such that $\tilde{a}_{\theta_j} \neq \tilde{a}_{\theta'_j}$. Assume without loss of generality that $\tilde{a}_{\theta_j} > \tilde{a}_{\theta'_j}$. We fix an agent $i \in S$ and two type profiles $\theta_{-S} \neq \theta'_{-S}$ such that θ_j is a component of θ_{-S} and θ'_j is a component of θ'_{-S}. Moreover, we fix a type profile $\theta_S \in \Theta_S$ that attains the maximum in this set $\{\frac{p(\tilde{\theta}_S, \theta_{-S})}{p(\tilde{\theta}_S, \theta'_{-S})} | \tilde{\theta}_S \in \Theta_S\}$. On each side of (17), by focusing on the dimensions corresponding to $(i, (\theta_S, \theta_{-S}))$, and $(j, (\theta_S, \theta_{-S}))$, we have

$$\tilde{b}_{(\theta_S, \theta_{-S})} = \tilde{a}_{\theta_i} p(\theta_S, \theta_{-S}) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,\ldots,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}) = \tilde{a}_{\theta_j} p(\theta_S, \theta_{-S}). \tag{20}$$

Similarly, by focusing on the dimensions corresponding to $(i, (\theta_S, \theta'_{-S}))$ and $(j, (\theta_S, \theta'_{-S}))$, we have

$$\tilde{b}_{(\theta_S, \theta'_{-S})} = \tilde{a}_{\theta_i} p(\theta_S, \theta'_{-S}) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,\ldots,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta'_{-S}) = \tilde{a}_{\theta'_j} p(\theta_S, \theta'_{-S}).$$

Therefore,

$$\frac{\tilde{a}_{\theta_j} p(\theta_S, \theta_{-S})}{\tilde{a}_{\theta'_j} p(\theta_S, \theta'_{-S})} = \frac{\tilde{a}_{\theta_i} p(\theta_S, \theta_{-S}) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S})}{\tilde{a}_{\theta_i} p(\theta_S, \theta'_{-S}) + \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta'_{-S})}$$
$$\in \text{con}\{\frac{p(\tilde{\theta}_S, \theta_{-S})}{p(\tilde{\theta}_S, \theta'_{-S})} | \tilde{\theta}_S \in \Theta_S\}.$$

However, the fact that $\tilde{a}_{\theta_j} > \tilde{a}_{\theta'_j}$ and that θ_S maximizes $\{\frac{p(\tilde{\theta}_S, \theta_{-S})}{p(\tilde{\theta}_S, \theta'_{-S})} | \tilde{\theta}_S \in \Theta_S\}$ imply

$$\frac{\tilde{a}_{\theta_j} p(\theta_S, \theta_{-S})}{\tilde{a}_{\theta'_j} p(\theta_S, \theta'_{-S})} > \frac{p(\theta_S, \theta_{-S})}{p(\theta_S, \theta'_{-S})} = \max\{\frac{p(\tilde{\theta}_S, \theta_{-S})}{p(\tilde{\theta}_S, \theta'_{-S})} | \tilde{\theta}_S \in \Theta_S\}.$$

The two observations yield a contradiction. To this end, we have established that there exists $\kappa' \in \mathbb{R}_{++}$ such that $\kappa' = \tilde{a}_{\theta_j}$ for all $\theta_j \in \Theta_j$.

For the above $j \notin S$ and any $i \in S$, by the argument to establish (20), we conclude that

$$(\kappa' - \kappa) p(\theta) = \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}), \forall \theta = (\theta_S, \theta_{-S}) \in \Theta.$$

Recall that all δ^{Sk} are non-truthful, and thus, there exists $k \in \{1, ..., \bar{k}\}$ and two different type profiles $\bar{\theta}_S \neq \theta_S$ such that $c_k \delta^{Sk}[\bar{\theta}_S](\theta_S) > 0$. As a result, the only way for the above equation to hold is $\kappa' - \kappa > 0$, which allows us to rewrite the above expression into

$$p(\theta) = \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} \frac{c_k}{\kappa' - \kappa} \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}), \forall \theta = (\theta_S, \theta_{-S}) \in \Theta. \tag{21}$$

Moreover, since each $c_k \geqslant 0$ with the strict inequality holds for at least one k,

$$\sum_{\theta \in \Theta} p(\theta) = 1 = \sum_{\theta \in \Theta} \sum_{\bar{\theta}_S \in \Theta_S} \sum_{k=1,...,\bar{k}} \frac{c_k}{\kappa' - \kappa} \delta^{Sk}[\bar{\theta}_S](\theta_S) p(\bar{\theta}_S, \theta_{-S}) = \sum_{k=1,...,\bar{k}} \frac{c_k}{\kappa' - \kappa}.$$

As a result, the vector $(\frac{c_1}{\kappa' - \kappa}, ..., \frac{c_{\bar{k}}}{\kappa' - \kappa})$ is in the simplex. Therefore, (21) contradicts with our observation from Step 1. □

Proof of Proposition 2. Statement 2 ⇒ Statement 1 follows from Lemma 6. It remains to establish Statement 1 ⇒ Statement 2.

Step 1. Fix any efficient allocation rule $q : \Theta \to A$, and construct a transfer rule η.

For each $i \in I$ and $\theta_i \in \Theta_i$, define

$$w_i(\theta_i) \equiv \sum_{\theta_{-i} \in \Theta_{-i}} [u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) + \frac{1}{n} u_0\big(q(\theta_i, \theta_{-i})\big)] p(\theta_{-i} | \theta_i) - \frac{1}{n} FS.$$

It is clear that $\sum_{i\in I}\sum_{\theta_i\in\Theta_i} w_i(\theta_i)p(\theta_i) = 0$. Hence, by Lemma A.3 of Kosenok and Severinov (2008), there exists an ex-post budget balanced transfer rule $\tau : \Theta \to \mathbb{R}^n$ such that $\sum_{\theta_{-i}\in\Theta_{-i}} \tau_i(\theta_i, \theta_{-i})p(\theta_{-i}|\theta_i) = w_i(\theta_i)$ for all $i \in I$ and $\theta_i \in \Theta_i$.

For each $i \in I$ and $\theta \in \Theta$, define $\eta_i(\theta) \equiv \frac{1}{n}u_0\big(q(\theta)\big) - \tau_i(\theta) - \frac{1}{n}FS$. Apparently,

$$\sum_{i\in I}\eta_i(\theta) = u_0\big(q(\theta)\big) - FS, \forall\theta \in \Theta. \tag{22}$$

Also, for all $i \in I$ and $\theta_i \in \Theta_i$,

$$\begin{aligned}\sum_{\theta_{-i}\in\Theta_{-i}} \eta_i(\theta_i, \theta_{-i})p(\theta_{-i}|\theta_i) &\overset{\text{definition of}\,\eta,\tau}{=} \frac{1}{n}\sum_{\theta_{-i}\in\Theta_{-i}} u_0\big(q(\theta_i, \theta_{-i})\big)p(\theta_{-i}|\theta_i) - w_i(\theta_i) - \frac{1}{n}FS\\ &\overset{\text{definition of}\,w}{=} -\sum_{\theta_{-i}\in\Theta_{-i}} u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big)p(\theta_{-i}|\theta_i).\end{aligned} \tag{23}$$

Step 2. Construct a multiplier λ_1.

For each $i \in I$ and $\bar{\theta}_i$, by Lemma 2, there exists $\phi^{\bar{\theta}_i}$ satisfying the conditions stated therein. Now, fix any $\lambda_1 \in \mathbb{R}_+$ that is weaky larger than

$$\max_{\substack{i\in I,\\ \bar{\theta}_i,\hat{\theta}_i\in\Theta_i \text{ with } \bar{\theta}_i\neq\hat{\theta}_i}} \frac{-V_i[q,\eta](\bar{\theta}_i,\hat{\theta}_i)}{\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^{\bar{\theta}_i}(\hat{\theta}_i,\theta_{-i})p(\theta_{-i}|\bar{\theta}_i)}.$$

Hence, for all $i \in I$, $\bar{\theta}_i$, $\hat{\theta}_i \in \Theta_i$ with $\bar{\theta}_i \neq \hat{\theta}_i$,

$$0 \geqslant V_i[q,\eta](\bar{\theta}_i,\hat{\theta}_i) + \lambda_1 \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^{\bar{\theta}_i}(\hat{\theta}_i,\theta_{-i})p(\theta_{-i}|\bar{\theta}_i).$$

Step 3. Construct a multiplier λ_2.

For each $S \in 2^I\backslash\{\varnothing, I\}$ with $2 \leqslant |S| \leqslant n-1$, let $\phi^S : \Theta \to \mathbb{R}^n$ be a transfer rule satisfying conditions stated in Lemma 3. Moreover, let $\delta^{S1}, ..., \delta^{S\bar{k}}$ denote all non-truthful deterministic joint reporting strategies. For each $k = 1, ..., \bar{k}$, let $\tilde{q}^k : \Theta \to A$ be any allocation rule satisfying (3) and (4) under δ^{Sk}. Now fix $\lambda_2 \in \mathbb{R}_+$ that strictly larger than

$$\max_{\substack{S\in2^I\backslash\{\varnothing,I\}\text{ with }2\leqslant|S|\leqslant n-1, k\in\{1,...,\bar{k}\},\\ \tilde{q}^k:\Theta\to A \text{ satisfying (3)(4) under } \delta^{Sk}}} \frac{-\sum_{\theta_S\in\Theta_S} V_S[\tilde{q}^k,\eta^{\delta^{Sk}}](\theta_S,\theta_S)p(\theta_S)}{\sum_{\theta_S\in\Theta_S}\sum_{\hat{\theta}_S\in\Theta_S}\sum_{i\in S}\sum_{\theta_{-S}\in\Theta_{-S}} \phi_i^S(\hat{\theta}_S,\theta_{-S})p(\theta_S,\theta_{-S})\delta^{Sk}[\theta_S](\hat{\theta}_S)}.$$

Hence, for all $S \in 2^I \backslash \{\varnothing, I\}$ with $2 \leqslant |S| \leqslant n-1$, deterministic $\delta^{Sk} \neq \bar{\delta}^S$, and $\tilde{q}^k : \Theta \to A$ satisfying (3) and (4) under δ^{Sk},

$$0 > \sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^k, \eta^{\delta^{Sk}}](\theta_S, \theta_S) p(\theta_S) + \lambda_2 \sum_{\theta_S \in \Theta_S} \sum_{\hat{\theta}_S \in \Theta_S} \sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i^S(\hat{\theta}_S, \theta_{-S}) p(\theta_S, \theta_{-S}) \delta^{Sk}[\theta_S](\hat{\theta}_S). \quad (24)$$

Step 4. When $n \geqslant 3$, define $T \equiv \{\eta + \lambda_1 \phi^{\tilde{\theta}_i} | i \in I, \tilde{\theta}_i \in \Theta_i\} \cup \{\eta + \lambda_2 \phi^S | S \in 2^I \backslash \{\varnothing, I\}$ with $2 \leqslant |S| \leqslant n-1\}$. When $n = 2$, define $T \equiv \{\eta + \lambda_1 \phi^{\tilde{\theta}_i} | i \in I, \tilde{\theta}_i \in \Theta_i\}$. Show that (q, T) is a feasible ambiguous mechanism such that the MD's ex-post revenue is constant and equal to FS.

We first show that (5) is satisfied. To see this, for each $t \in T$, either there is $i \in I$ and $\tilde{\theta}_i \in \Theta_i$ such that $t = \eta + \lambda_1 \phi^{\tilde{\theta}_i}$ or there is S such that $t = \eta + \lambda_2 \phi^S$. Notice that $\phi^{\tilde{\theta}_i}$ and ϕ^S are ex-post budget balanced. As a result,

$$u_0\big(q(\theta)\big) - \sum_{i \in I} t_i(\theta) = u_0\big(q(\theta)\big) - \sum_{i \in I} \eta_i(\theta) \overset{(22)}{=} u_0\big(q(\theta)\big) - [u_0\big(q(\theta)\big) - FS] = FS$$

for all $\theta \in \Theta$, i.e., the ex-post payoff of the MD is constant and equal to FS.

Then we show that $V_i[q, T](\theta_i, \theta_i) = 0$ for all $i \in I$ and $\theta_i \in \Theta_i$, i.e., IR binds. To see this, for each $t \in T$, either there is $i \in I$ and $\tilde{\theta}_i \in \Theta_i$ such that $t = \eta + \lambda_1 \phi^{\tilde{\theta}_i}$ or there is S such that $t = \eta + \lambda_2 \phi^S$. Recall Condition (i) in Lemma 2 and Condition (i) in Lemma 3. For each $t \in T$, $i \in I$, and $\theta_i \in \Theta_i$,

$$V_i[q, t](\theta_i, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} [u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) + \eta_i(\theta_i, \theta_{-i})] p(\theta_{-i} | \theta_i) \overset{(23)}{=} 0. \quad (25)$$

To demonstrate IC, for each $i \in I$, $\theta_i \in \Theta_i$, $\hat{\theta}_i \in \Theta_i \backslash \{\theta_i\}$, and $t = \eta + \lambda_1 \phi^{\theta_i} \in T$,

$$V_i[q, T](\theta_i, \theta_i) = 0 \geqslant V_i[q, \eta](\theta_i, \hat{\theta}_i) + \lambda_1 \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i^{\theta_i}(\hat{\theta}_i, \theta_{-i}) p(\theta_{-i} | \theta_i) = V_i[q, t](\theta_i, \hat{\theta}_i),$$

where the inequality follows from the choice of λ_1. The above two expressions jointly imply that $0 \geqslant V_i[q, t](\theta_i, \sigma_i)$ for all $i \in I$ and reporting strategy σ_i. Therefore, $V_i[q, T](\theta_i, \theta_i) \geqslant V_i[q, T](\theta_i, \sigma_i)$ for any σ_i.

To this end, we have completed Step 4.

Step 5. Show that (q, T) satisfies RCP*.

Every I-feasible ambiguous I-reallocational manipulation leads to a feasible ambiguous mechanism because there is no agent out of I. Also, notice that the MD's ex-post payoff is constant and equal to FS. Hence, (q, T) satisfies RCP* with respect to I.

To show that (q,T) satisfies RCP* with respect to each non-grand coalition S, suppose that an ambiguous S-collusive mechanism (δ^S, Ψ^S) induces an S-feasible ambiguous S-reallocational manipulation $(\tilde{q}, T^{\delta^S} + \Psi^S)$. When $\delta^S = \bar{\delta}^S$, by (3) and (4) as well as the feasibility of (q,T), this manipulation leads to a feasible ambiguous mechanism. Also, this manipulation does not affect the MD's payoff. Hence, it only remains to discuss the case where $\delta^S \neq \bar{\delta}^S$.

As $\delta^S \neq \bar{\delta}^S$ can be non-deterministic, view δ^S as a lottery $\delta^S = \beta_0\bar{\delta}^S + \sum_{k=1,...,\bar{k}} \beta_k \delta^{Sk}$, where each δ^{Sk} is a non-truthful deterministic joint reporting strategy, $(\beta_0, ..., \beta_{\bar{k}})$ is in the simplex, and there exists $k \in \{1, 2, ..., \bar{k}\}$ such that $\beta_k > 0$. Also, view $\tilde{q}$ as a lottery $\tilde{q} = \beta_0 \tilde{q}^0 + \sum_{k=1,...,\bar{k}} \beta_k \tilde{q}^k$, where $\tilde{q}^0 : \Theta \to A$ satisfies expressions (3) and (4) under $\bar{\delta}^S$, and for each $k \in \{1, ..., \bar{k}\}$, $\tilde{q}^k : \Theta \to A$ satisfies (3) and (4) under δ^{Sk}.

Let $t = \eta + \lambda_2\phi^S \in T$ now. It is useful to notice that

$$\sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^0, t^{\bar{\delta}^S}](\theta_S, \theta_S)p(\theta_S) = \sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^0, t](\theta_S, \theta_S)p(\theta_S) \leqslant \sum_{\theta_S \in \Theta_S} V_S[q, t](\theta_S, \theta_S)p(\theta_S) \overset{(25)}{=} 0.$$

The first equality above follows from the fact that $\bar{\delta}^S$ is truthful. The inequality follows from the fact that $\bar{\delta}^S$ is truthful in expressions (3) and (4) as well as the efficiency of q. Also, recall from (24), for each $k \in \{1, 2, ..., \bar{k}\}$,

$$\sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^k, t^{\delta^{Sk}}](\theta_S, \theta_S)p(\theta_S) < 0.$$

These inequalities as well as the fact that there exists $k \in \{1, ..., \bar{k}\}$ such that $\beta_k > 0$ lead to

$$\begin{aligned} &\sum_{\theta_S \in \Theta_S} V_S[\tilde{q}, t^{\delta^S}](\theta_S, \theta_S)p(\theta_S) \\ =&\beta_0 \sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^0, t^{\bar{\delta}^S}](\theta_S, \theta_S)p(\theta_S) + \sum_{k=1,...,\bar{k}} \beta_k \sum_{\theta_S \in \Theta_S} V_S[\tilde{q}^k, t^{\delta^{Sk}}](\theta_S, \theta_S)p(\theta_S) < 0. \end{aligned} \tag{26}$$

Now fix any $\psi^S \in \Psi^S$. By S-IR, for all $i \in S$, $\theta_i \in \Theta_i$, and $t = \eta + \lambda_2\phi^S \in T$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} \left[u_i\big(\tilde{q}(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) + \sum_{\theta'_S \in \Theta_S} t_i(\theta'_S, \theta_{-S})\delta^S[\theta_S](\theta'_S) + \psi_i^S\big(t, (\theta_i, \theta_{-i})\big)\right]p(\theta_{-i}|\theta_i) \geqslant 0.$$

A weighted sum of the above inequalities and the budget balance of ψ^S within S imply that

$$\sum_{\theta_S \in \Theta_S} V_S[\tilde{q}, t^{\delta^S}](\theta_S, \theta_S)p(\theta_S) \geqslant 0,$$

which contradicts (26). To this end, we conclude that (q,T) satisfies RCP* with respect to S. □