

# Mechanism Design with Ambiguous Transfers: An Analysis in Finite Dimensional Naive Type Spaces

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June 4, 2019

## Abstract

This paper introduces ambiguous transfers to the problems of full surplus extraction and implementation in finite dimensional naive type spaces. The mechanism designer commits to one transfer rule but informs agents of a set of potential ones. Without knowing the adopted transfer rule, agents are assumed to make decisions based on the worst-case expected payoffs. A key condition in this paper is the Beliefs Determine Preferences (BDP) property, which requires an agent to hold distinct beliefs about others' information under different types. We show that full surplus extraction can be guaranteed via ambiguous transfers if and only if the BDP property is satisfied by all agents. When agents' beliefs can be generated by a common prior, all efficient allocations are implementable via individually rational and budget-balanced mechanisms with ambiguous transfers if and only if the BDP property holds for all agents. This necessary and sufficient condition is weaker than those for full surplus extraction and implementation via Bayesian mechanisms. Therefore, ambiguous transfers may achieve first-best outcomes that are impossible under the standard approach. In particular, with ambiguous transfers, efficient allocations become implementable generically in two-agent problems, a result that does not hold under a Bayesian framework.

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I am indebted to Nicholas Yannelis for his guidance and encouragement. I also thank Rabah Amir, Colin Campbell, Oriol Carbonell-Nicolau, Chris Chambers, Yi-Chun Chen, Yifan Dai, Inga Deimen, Luciano de Castro, Hülya Eraslan, Wei He, Toomas Hinnosaar, R. Vijay Krishna, Jin Yeub Kim, Jiangtao Li, Heng Liu, Rich McLean, Malleesh Pai, Marilyn Pease, Laura Razzolini, Tomas Sjöström, Barry Sopher, Max Stinchcombe, Rodrigo Velez, and seminar participants at the University of Iowa, Rutgers University, Shanghai Jiao Tong University, Tsinghua University, Texas A&M University, Cornell University, Florida State University, 2017 Asian Meeting of the Econometric Society, 2017 China Meeting of the Econometric Society, 2017 Midwest Economic Theory Conference, 2018 Texas Economic Theory Camp, and 2019 North American Winter Meeting of the Econometric Society for valuable discussions. I also wish to express my gratitude to an associate editor and a referee for helpful comments and suggestions.

**Keywords:**JEL classification: C72; D81; D82

**Keywords:** Full surplus extraction; Bayesian (partial) implementation; Ambiguous transfers; Correlated beliefs; Individual rationality; Budget balance

## 1 Introduction

Many transaction mechanisms have uncertain rules. For instance, Priceline Express Deals offer travelers a known price for a hotel stay, but the exact name and location of the hotel remain unknown until the completion of payment. Alternatively, some stores run scratch-and-save promotions. Consumers receive scratch cards during check-out, which reveal discounts of uncertain levels, and thus the costs of their purchases remain unknown at the time they decide to buy. As a third example, eBay allows sellers of auction-style listings to set hidden reserve prices.

In all the above mechanisms, the mechanism designer introduces uncertainty about the allocation and/or transfer rule without telling agents the underlying probability distribution. The subjective expected utility model can be adopted to describe agents' decision making without an objective probability. However, since Ellsberg (1961), many studies have challenged this model, arguing that decision makers tend to be ambiguity-averse. Therefore, it is important to understand if and how a mechanism designer can benefit from agents' ambiguity aversion. More specifically, we would like to know whether engineering ambiguity on rules of mechanisms can help the designer achieve the first-best outcome.

This paper introduces ambiguous transfers to study two problems. One is full surplus extraction. The other is interim individually rational and ex-post budget-balanced implementation of any ex-post efficient allocation rule. There is one mechanism designer (assumed to be female) and at least two agents (assumed to be male). The analysis is based on finite dimensional naive type spaces, i.e., those in which each agent's type is his payoff-relevant private information. The problem of full surplus extraction aims to design a mechanism in which agents transfer the entire surplus to the designer. The efficient implementation problem constructs an interim incentive compatible, interim individually rational, and ex-post budget-balanced mechanism such that the socially optimal outcome emerges as an equilibrium. In our model, the mechanism designer informs agents of the exact allocation rule. She also commits to one transfer rule, but the communication is ambiguous so that agents only know a set of potential transfer rules. Without knowing the adopted transfer rule, agents are assumed to be ambiguity-averse. More specifically, agents are maxmin expected utility maximizers who make decisions based on the worst-case scenario.

Our main result shows that the necessary and sufficient condition to ensure the existence of first-best mechanisms with ambiguous transfers is the Beliefs Determine Preferences (BDP) property.<sup>1</sup> In particular, we show that full surplus extraction can be guaranteed via ambiguous transfers if and only if the BDP property is satisfied by all agents. In addition, when agents' beliefs can be generated by a common prior, every efficient allocation rule is implementable via an interim individually rational and ex-post budget-balanced mechanism with ambiguous transfers if and only if the BDP property holds for all agents. By confining the analysis to private value common prior environments, we further show that efficient implementation can be guaranteed if and only if the BDP property fails for at most one agent. As an extension, we establish necessary and slightly stronger sufficient conditions for efficient implementation when beliefs may not be generated by a common prior. Lastly, we discuss the robustness of our sufficiency results under alternative models of ambiguity aversion.

The BDP property is weaker than Crémer and McLean (1988)'s Convex Independence condition, which is necessary and sufficient to guarantee full surplus extraction via Bayesian mechanisms. Convex Independence, together with the Identifiability condition established by Kosenok and Severinov (2008), is necessary and sufficient for implementing any efficient allocation rule via an interim individually rational and ex-post budget-balanced Bayesian mechanism. In both the full surplus extraction and the implementation problems, the ambiguous mechanism design approach requires a strictly weaker condition to obtain the first-best outcome than the Bayesian approach. As a result, when the conditions of Crémer and McLean (1988) or Kosenok and Severinov (2008) fail, engineering ambiguity deliberately may allow the designer to achieve first-best outcomes that are impossible under Bayesian mechanisms. Intuitively, this is because when ambiguous transfers are introduced, we do not need to construct one transfer rule satisfying all incentive compatibility constraints simultaneously. Instead, different transfer rules can be adopted to guarantee different incentive compatibility constraints, which may make mechanism design problems easier.

In applications, it is of interest to study some cases where Crémer and McLean (1988) or Kosenok and Severinov (2008)'s necessary and sufficient conditions fail, although a few works that we discuss in Section 1.1 show that these conditions are weak in some sense. In particular, the BDP property imposes weaker restrictions on the cardinality of the finite type space than Convex Independence and Identifiability. For example, when one agent has more types than the number of type profiles of all other agents, Convex Independence fails for this agent with positive probability, in which case the mechanism designer cannot always extract the full surplus. As another instance, Kosenok and Severinov (2008)'s necessary

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<sup>1</sup>The property was introduced by Neeman (2004). It requires that an agent should hold distinct beliefs about others' types under different types of himself.

and sufficient conditions can never hold simultaneously for any common prior with only two agents, indicating an impossibility result on two-agent implementation problems. However, the BDP property holds for all agents in finite dimensional naive type spaces generically regardless of the number of agents and the cardinality of the type space. Hence, allowing for ambiguous transfers may help the mechanism designer to resolve these impossibility results.

In this paper, the mechanism designer announces an efficient allocation rule and introduces ambiguity in transfer rules only. As the ex-post efficient allocation rule is often unique in a finite-type framework, the mechanism designer may not have multiple allocation rules to choose from. In a related paper, Di Tillio et al. (2017) study how second-best revenue in an independent private value auction can be improved if the seller introduces ambiguity in both allocation and transfer rules. We discuss more on the relationship with this paper in Section 1.1.

The paper proceeds as follows. We review the literature in Section 1.1 and introduce the environment in Section 2. Our main result is presented in Section 3. Section 4 extends our main result along two directions. Section 5 concludes. The Appendix collects all omitted proofs from Section 3. Omitted proofs and examples from Section 4 are relegated to the Online Appendices.

## **1.1 Literature review**

### **1.1.1 Efficient mechanism design**

How to implement efficient allocations is a classical topic in mechanism design theory that has been widely studied in situations such as public good provision and bilateral trading. Individual rationality is a natural requirement as agents can opt out of the mechanism. Budget balance requires that agents should finance within themselves for the efficient outcome rather than rely on an outside budget-breaker. When either individual rationality or budget balance is required, the literature provides positive results for efficient mechanism design in private value environments. For instance, the VCG mechanism (Vickrey (1961), Clarke (1971), and Groves (1973)) is ex-post individually rational. The AGV mechanism (d'Aspremont and Gérard-Varet (1979)) is ex-post budget-balanced. However, the literature documents a tension between efficiency, individual rationality, and budget balance, when agents have independent information. For example, in a private value bilateral trading framework, Myerson and Satterthwaite (1983) prove that it is impossible to achieve efficiency with an individually rational and budget-balanced mechanism in general. With multi-dimensional and interdependent values, Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) prove that efficient allocations are generically non-implementable.

First-best mechanism design becomes possible in some correlated information environments. Crémer and McLean (1985, 1988) establish two conditions to fully extract agents' surplus in private value auctions, among which the Convex Independence condition is necessary and sufficient for full surplus extraction to be a Bayesian Nash equilibrium. When the type space has finite dimensions, if no agent has more types than all others' type profiles, the condition holds for all agents under almost all profiles of beliefs. Without restricting the dimension, different notions of genericity are adopted in the literature and various conclusions on genericity of Convex Independence (or the weaker BDP property) are made (e.g., Neeman (2004), Heifetz and Neeman (2006), Barelli (2009), Chen and Xiong (2011, 2013), Gizatulina and Hellwig (2014, 2017)). With continuous types, McAfee and Reny (1992) show that approximate full surplus extraction can be achieved. In addition, the recent papers of Liu (2018) and Noda (2019) prove an intertemporal variant of Convex Independence is sufficient for first-best mechanism design in dynamic environments. By introducing ambiguous transfers, Section 3 of the current paper shows that a weaker condition, the BDP property, becomes necessary and sufficient for full surplus extraction.

In an implementation problem, the allocation rule is exogenously given. Thus, the mechanism designer constructs incentive compatible transfers to achieve the desired outcome. In the context of exchange economies, McLean and Postlewaite (2002, 2003a,b) propose the notion of informational size and prove the existence of incentive compatible and approximately efficient outcomes when agents have small informational size.<sup>2</sup> Under a mechanism design framework, McLean and Postlewaite (2004, 2015) implement efficient allocation rules via individually rational mechanisms under the BDP property. In their mechanisms, small outside money is needed even when agents are informationally small. Different from these papers, our mechanism for implementation in Section 3 is exactly efficient, individually rational, and budget-balanced without imposing any informational smallness assumption.

A few papers study budget-balanced mechanisms with or without independent information, including Matsushima (1991), Aoyagi (1998), Chung (1999), d'Aspremont et al. (2004), Miller et al. (2007), etc.<sup>3</sup> Among these works, d'Aspremont et al. (2004) propose necessary and sufficient conditions for budget-balanced mechanisms. None of these papers can guarantee individual rationality. Also, they assume that there are at least three agents. However, we are able to obtain individual rationality in addition to budget balance, and our mechanism with ambiguous transfers works for environments even when there are only two agents.

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<sup>2</sup>For related results, see also Sun and Yannelis (2007, 2008).

<sup>3</sup> Matsushima (1991), Chung (1999), d'Aspremont et al. (2004) only consider private value utility functions. In this case, incentive compatibility can be achieved via a VCG mechanism, rather than via information correlation. Thus, they allow for independent information.

Matsushima (2007), Kosenok and Severinov (2008), and Gizatulina and Hellwig (2010) among others design individually rational and budget-balanced mechanisms. Kosenok and Severinov (2008) propose the Identifiability condition, which along with the Convex Independence condition, is necessary and sufficient for implementing any ex-ante socially rational allocation rule via an interim individually rational and ex-post budget-balanced Bayesian mechanism. The Identifiability condition holds for almost all common priors with at least three agents and under some restrictions on the dimension of the type space, but Convex Independence and Identifiability never hold simultaneously in a two-agent setting. Thus, Kosenok and Severinov (2008) imply an impossibility result in efficient, individually rational, and budget-balanced two-agent mechanism design. The BDP property is weaker than Convex Independence, and the Identifiability condition is relaxed. Moreover, the BDP property holds generically in our environments even when there are only two agents, and thus we make the impossible possible for two-agent implementation problems.

### 1.1.2 Mechanism design under ambiguity

In the growing literature on mechanism design with ambiguity-averse agents, most of the works assume exogenously that agents hold ambiguous beliefs of others' types. For example, Bose et al. (2006), Bose and Daripa (2009), and Bodoh-Creed (2012) study optimal mechanism design with ambiguity-averse agents. De Castro and Yannelis (2018) and De Castro et al. (2017a,b) prove that all Pareto efficient allocations are incentive compatible and thus implementable when agents' ambiguous beliefs are unrestricted. Under the private value assumption, Wolitzky (2016) establishes a necessary condition for the existence of an efficient, individually rational, and weak budget-balanced mechanism. In an environment with multi-dimensional and interdependent values, Song (2018) quantifies the amount of ambiguity that is necessary and sometimes sufficient for efficient mechanism design. We do not assume exogenous ambiguity in agents' beliefs, which is the biggest difference from the above papers.

Bose and Renou (2014) and Di Tillio et al. (2017) contrast the above works in that ambiguity is endogenously engineered by the mechanism designer. Before the allocation stage, Bose and Renou (2014) let the mechanism designer communicate with agents via an ambiguous device, which generates multiple beliefs. Their paper characterizes social choice functions that are implementable under this method. Our paper is different from Bose and Renou (2014), as we do not need multiple beliefs on other agents' private information.

Di Tillio et al. (2017) consider the problem of revenue maximization in a private value and independent belief auction. The seller commits to a simple mechanism, i.e., an allocation and

transfer rule, but informs agents of a set of simple mechanisms. As all the simple mechanisms generate the same expected revenue (imposed by the Consistency condition), agents do not know the exact rule and thus make decisions based on the worst-case scenario. Compared to the Bayesian mechanism, their ambiguous approach yields a higher expected revenue.

In the current paper, ambiguity is engineered in a similar way to Di Tillio et al. (2017). However, instead of studying how ambiguous mechanisms improve second-best revenues under independent beliefs, our paper studies when the first-best outcome in surplus extraction or implementation can be achieved without restricting attention to independent beliefs. The essential factor that enables us to achieve the first-best outcome in a finite type space is the correlation in agents' beliefs and more particularly, the BDP property.

As mentioned before, we fix an efficient allocation rule and only allow for ambiguity in transfer rules, but in Di Tillio et al. (2017)'s mechanism both allocation and transfer rules are ambiguous. Our restriction on unambiguous allocation rule is compatible with Di Tillio et al. (2017)'s Consistency condition. In the full surplus extraction problem, each transfer rule leaves agents zero surplus and gives the designer the full surplus on path. In the efficient implementation problem, each transfer rule leads to the first-best efficiency on path. Therefore, all transfer rules are credible. The restriction on unambiguous allocation rule is closely related to two facts: first, we aim to achieve the first-best outcome in full surplus extraction or implementation, and second, our argument is confined to finite type spaces. Allowing for ambiguity in allocation rules may fail full surplus extraction and implementation. To see this, consider a finite-type environment where the total surplus is maximized by a unique allocation rule. In this case, any other allocation rule is inefficient and has a lower surplus level. As the efficient allocation rule must be used in the mechanisms for full surplus extraction and implementation, and as agents know the designer's objective is to extract the full surplus or maximize efficiency, any other rule with a lower surplus level is non-credible to the agents. Hence, multiple allocation rules are not used in our environment.

In Di Tillio et al. (2017)'s optimal mechanism under independent beliefs and finitely many types, ambiguity in allocation rules plays a role. Therefore, they cannot obtain the first-best revenue. In fact, in a screening or an independent private value auction framework, allowing for ambiguous transfers but not ambiguous allocations does not improve the seller's revenue compared to a standard unambiguous mechanism. However, according to Di Tillio et al. (2017)'s Appendix B, their approach works for full surplus extraction with continuous types. This is because there are infinitely many ex-ante efficient allocation rules. Among them, every two rules are the same except on a null set in the type space. With continuous types, if an efficiency-maximizing social planner wants to implement an ex-ante efficient allocation rule, she can follow the approach of Di Tillio et al. (2017)'s Appendix B as well. Hence, the

current paper focuses on environments with finitely many types.

## 2 Environment

We study an asymmetric information environment given by  $\mathcal{E} = \{I, A, (\Theta_i, u_i, p_i)_{i=1}^N\}$ . Let  $I = \{1, \dots, N\}$  be a finite set of **agents**. Assume  $N \geq 2$ . Denote the set of **feasible outcomes** by  $A$ . Let  $\theta_i \in \Theta_i$  be agent  $i$ 's **type**, which is his payoff-relevant private information. Denote the **type space**  $\times_{i \in I} \Theta_i$  by  $\Theta$ . For simplicity, denote  $\times_{j \in I, j \neq i} \Theta_j$  by  $\Theta_{-i}$  and  $\times_{k \in I, k \neq i, j} \Theta_k$  by  $\Theta_{-i-j}$ . Let  $|\Theta_i|$  be the cardinality of  $\Theta_i$ . Assume  $2 \leq |\Theta_i| < \infty$ . Each agent  $i$  has a quasi-linear **utility function**  $u_i(a, \theta) + b$ , where  $a \in A$  is a feasible outcome,  $b \in \mathbb{R}$  is a monetary transfer, and  $\theta \in \Theta$  is the realized type profile. For each  $\theta_i \in \Theta_i$ , let  $p_i(\cdot | \theta_i) \in \Delta(\Theta_{-i})$  be type- $\theta_i$  agent  $i$ 's **belief** of other agents' types, where type- $\theta_i$  agent  $i$  believes that others have type profile  $\theta_{-i}$  with probability  $p_i(\theta_{-i} | \theta_i)$ .

The structure of environment  $\mathcal{E}$  is assumed to be common knowledge between the mechanism designer and agents, but every agent's realized type is his private information. As a type in this paper only concerns payoff-relevant information, such a type space is sometimes called a naive type space in the literature.

We impose the following assumption on beliefs throughout the paper.

**Assumption 2.1:** For all  $i, j \in I$  with  $i \neq j$ , and  $(\theta_i, \theta_j) \in \Theta_i \times \Theta_j$ , type- $\theta_i$  agent  $i$ 's marginal belief that agent  $j$  has type  $\theta_j$  is positive, i.e.,  $p_i(\theta_j | \theta_i) \equiv \sum_{\theta_{-i-j}} p_i(\theta_j, \theta_{-i-j} | \theta_i) > 0$ .

We remark that when  $N \geq 3$ , Assumption 2.1 is weaker than the full support assumption, which requires instead that  $p_i(\theta_{-i} | \theta_i) > 0$  for each  $i$ ,  $\theta_i$ , and  $\theta_{-i}$ .

Two more conditions will be imposed for a fraction of our later analysis. In the special case that  $u_i(a, (\theta_i, \theta_{-i})) = u_i(a, (\theta_i, \theta'_{-i}))$  for all  $\theta_i \in \Theta_i$ ,  $\theta_{-i}, \theta'_{-i} \in \Theta_{-i}$ , and  $a \in A$ , we say  $u_i$  has **private value** and denote  $u_i(a, (\theta_i, \theta_{-i}))$  by  $u_i(a, \theta_i)$ . We say a profile of beliefs  $(p_i)_{i \in I}$  can be generated by the **common prior**  $p \in \Delta(\Theta)$  if the marginal probability  $p(\theta_i) \equiv \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i, \theta_{-i}) > 0$  and the conditional probability  $p(\theta_{-i} | \theta_i) \equiv \frac{p(\theta)}{p(\theta_i)} = p_i(\theta_{-i} | \theta_i)$  for all  $i \in I$ ,  $\theta_i \in \Theta_i$ , and  $\theta_{-i} \in \Theta_{-i}$ . A profile of beliefs  $(p_i)_{i \in I}$  is said to satisfy the **common prior assumption** if there exists  $p \in \Delta(\Theta)$  such that  $(p_i)_{i \in I}$  can be generated by the common prior  $p$ . We will be explicit when imposing any of the conditions in later analysis.

An **allocation rule**  $q : \Theta \rightarrow A$  is a plan to assign a feasible outcome contingent on agents' realized type profile. An allocation rule  $q$  is said to be ex-post **efficient** if  $\sum_{i \in I} u_i(q(\theta), \theta) \geq \sum_{i \in I} u_i(a, \theta)$  for all  $a \in A$  and  $\theta \in \Theta$ .

**Definition 2.1:** A *mechanism with ambiguous transfers* is a pair  $\mathcal{M} = (q, \Phi)$ , where



$q : \Theta \rightarrow A$  is an allocation rule, and  $\Phi$  is a set of transfer rules with a generic element  $\phi : M \rightarrow \mathbb{R}^N$ . We call the set  $\Phi$  **ambiguous transfers**.<sup>4</sup>

The mechanism works in the following way. The designer first commits to the allocation rule  $q : M \rightarrow A$  and an arbitrary transfer rule  $\phi \in \Phi$  secretly. Before reporting messages, agents are informed of the allocation rule  $q$  and ambiguous transfers  $\Phi$ , but not  $\phi$ . After agents report their messages, the mechanism designer reveals  $\phi$ . Then allocations and transfers are made according to the reported messages as well as  $q$  and  $\phi$ .

As agents only know the set  $\Phi$ , we follow the spirit of Gilboa and Schmeidler (1989)'s **maxmin expected utility** (MEU) and assume that agents make decisions based on the worst-case expected payoff. Hence, a type- $\theta_i$  agent  $i$ 's interim payoff is

$$\inf_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \phi_i(\theta_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i).$$

We remark that this expression follows Di Tillio et al. (2017) in adopting the infimum notation, since  $\Phi$  does not have to be compact. However, as we focus on finite dimensional type spaces, we are able to construct a finite set  $\Phi$  to fulfill our goal when the sufficient conditions in Theorem 3.1 or Proposition 4.1 are satisfied. We also remark that an agent  $i$  only cares about the transfer to himself,  $\phi_i$ . Thus, in the above expression, it is equivalent to let the infimum be chosen among all  $\phi_i \in \Phi_i$ .

Throughout this paper, the outside option  $x_0$  is normalized to give all agents zero payoffs at all type profiles. A mechanism with ambiguous transfers  $(q, \Phi)$  is said to satisfy interim **individual rationality** (IR) if  $\inf_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \phi_i(\theta_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i) \geq 0$  for all  $i \in I$  and  $\theta_i \in \Theta_i$ . It satisfies ex-post **budget balance** (BB) if  $\sum_{i \in I} \phi_i(\theta) = 0$  for all  $\phi \in \Phi$  and  $\theta \in \Theta$ . The mechanism is said to satisfy interim **incentive compatibility** (IC) if  $\inf_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \phi_i(\theta_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i) \geq \inf_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \phi_i(\theta'_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i)$  for all  $i \in I$  and  $\theta_i, \theta'_i \in \Theta_i$ .<sup>5</sup>

This paper studies two related but different objectives. One is full surplus extraction, and the other is implementation of an efficient allocation rule via an IR and BB mechanism.

In the sense of McAfee and Reny (1992), a mechanism with ambiguous transfers  $\mathcal{M} =$

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<sup>4</sup>We focus on direct mechanisms. One can follow Di Tillio et al. (2017) to establish a revelation principle, and thus the restriction on direct mechanisms is without loss of generality.

<sup>5</sup>Like many mechanism design works with ambiguity aversion, e.g., Wolitzky (2016), Di Tillio et al. (2017), Song (2018), we restrict attention to pure strategies. Depending on how the payoff of playing a mixed strategy is formalized, the restriction could be with or without loss of generality. See Wolitzky (2016) for more details.

$(q, \Phi)$  is said to **extract the full surplus** if it is IR and IC,  $q$  is ex-post efficient, and

$$\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta), \theta) + \phi_i(\theta)] p_i(\theta_{-i} | \theta_i) = 0, \forall \phi \in \Phi.$$

The requirement that every  $\phi \in \Phi$  extracts the full surplus follows from Di Tillio et al. (2017)'s Consistency condition. In other words, since the designer's objective is to extract the full surplus, any transfer rule that leaves agents a positive surplus is not credible.

When studying the implementation problem, we want the mechanism to be IR and BB so that agents are willing to participate and outside money is not needed to finance the efficient outcome. An allocation rule  $q$  is **implementable** by an IR and BB mechanism with ambiguous transfers if there exists an IC, IR, and BB mechanism with ambiguous transfers  $\mathcal{M} = (q, \Phi)$ .

### 3 Main result

Our key condition, the Beliefs Determine Preferences property, is introduced by Neeman (2004). When the BDP property holds for an agent, he should have distinct beliefs under different types of himself. Thus, it is necessary that agents' beliefs are correlated.

**Definition 3.1:** *The **Beliefs Determine Preferences** (BDP) property holds for agent  $i$  if there do not exist types  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$  such that  $p_i(\cdot | \bar{\theta}_i) = p_i(\cdot | \hat{\theta}_i)$ .*

The following theorem is the main result of the paper.

**Theorem 3.1:**

1. *Given a profile of beliefs  $(p_i)_{i \in I}$ , full surplus extraction under any profile of utility functions can be achieved via a mechanism with ambiguous transfers if and only if the BDP property holds for all agents;*
2. *when agents' beliefs  $(p_i)_{i \in I}$  can be generated by a common prior  $p$ , any ex-post efficient allocation rule under any profile of utility functions is implementable via an IR and BB mechanism with ambiguous transfers if and only if the BDP property holds for all agents;*
3. *when agents' beliefs  $(p_i)_{i \in I}$  can be generated by a common prior  $p$ , any ex-post efficient allocation rule under any profile of **private value** utility functions is implementable via an IR and BB mechanism with ambiguous transfers if and only if the BDP property holds for at least  $N - 1$  agents.*

We remark that the number of agents, the dimension of the finite type space, whether the utility functions have private or interdependent value, and whether agents' beliefs can

be generated by a common prior do not change the conclusion of Part 1 of the theorem.

Parts 2 and 3 of Theorem 3.1 focus on implementation via an IR and BB mechanism with ambiguous transfers, but full surplus extraction does not require the BB condition. To guarantee the BB condition and obtain unified necessary and sufficient conditions for implementation, we impose that beliefs can be generated by a common prior for Parts 2 and 3.<sup>6</sup> Part 2 does not restrict the environment to be a private value one while Part 3 does. When focusing on private value utility functions, Part 3 obtains a weaker condition for implementation compared to Part 2. However, according to Part 3, even if ambiguous transfers are allowed and we confine our analysis to private value environments, we can always find non-implementable efficient allocation rules under independent beliefs.

To prove the necessity half of the first two statements, when the BDP property fails for one agent, we construct a profile of utility functions under which full surplus extraction fails and an efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers.<sup>7</sup> When the BDP property fails for two agents, we construct private value utility functions under which efficient implementation fails.

We prove the sufficiency statements of Theorem 3.1 by constructing mechanisms consisting of two transfer rules. Although there are mechanisms with more transfers to extract the full surplus or implement the efficient allocation rule, to be consistent with the spirit of minimal mechanisms of Di Tillio et al. (2017), we only present the ones with two rules.

To prove the sufficiency direction of Part 1, the Appendix begins with several lemmas. Lemma A.1 shows that for each  $i \in I$  and  $\bar{\theta}_i \neq \hat{\theta}_i$ , there exists a transfer rule (a lottery)  $\psi^{\bar{\theta}_i, \hat{\theta}_i}$  such that (1) when all agents truthfully report, for every agent  $j \in I$  and type  $\theta_j \in \Theta_j$ , agent  $j$ 's component of lottery  $\psi^{\bar{\theta}_i, \hat{\theta}_i}$  gives type- $\theta_j$  agent  $j$  zero expected value, (2) agent  $i$ 's component of  $\psi^{\bar{\theta}_i, \hat{\theta}_i}$  gives type- $\bar{\theta}_i$  agent  $i$  a negative expected value when he unilaterally misreports  $\hat{\theta}_i$ . This step is proven via Fredholm's theorem of the alternative.

Lemmas A.2 and A.3 construct a linear combination of transfer rules  $(\psi^{\bar{\theta}_i, \hat{\theta}_i})_{i \in I, \bar{\theta}_i, \hat{\theta}_i \in \Theta_i, \bar{\theta}_i \neq \hat{\theta}_i}$ , denoted by  $\psi$ , such that (1) when all agents truthfully report, for every  $i$  and  $\theta_i$ , agent  $i$ 's component of lottery  $\psi$  gives type- $\theta_i$  agent  $i$  zero expected value, (2) for every  $i$  and  $\bar{\theta}_i \neq \hat{\theta}_i$ , when all other agents truthfully report, type- $\bar{\theta}_i$  agent  $i$  receives non-zero expected value when

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<sup>6</sup>The common prior assumption is used explicitly in Lemmas A.4 and A.5 and thus proof of the sufficiency direction of Parts 2 and 3 as well as the necessity direction of Part 3. Section 4.1 relaxes this assumption and presents necessary and (stronger) sufficient conditions on implementation.

<sup>7</sup>The construction adopts interdependent value utility functions so that there is a unified necessity proof for Parts 1 and 2. One can also follow Crémer and McLean (1988) to construct private value utility functions for the necessity proof of Part 1. By Part 3 of Theorem 3.1, when the BDP property fails for only one agent, efficient allocations under private value environments are implementable. Thus, the necessity direction of Part 2 has to be proved with interdependent value utility functions.

he unilaterally misreports  $\hat{\theta}_i$ .

Then pick an ex-post efficient allocation rule  $q$  and let  $\eta_i(\theta) = -u_i(q(\theta), \theta)$  for all  $i \in I$  and  $\theta \in \Theta$ . Let the set of ambiguous transfers for agent  $i$  be  $\Phi_i = \{\eta_i + c\psi_i, \eta_i - c\psi_i\}$ . Because  $\eta_i$  transfers agent  $i$ 's entire surplus to the mechanism designer, and because  $\psi_i$  has zero expected value when every agent truthfully reports, each IR constraint binds. In addition, as  $\psi_i$  has non-zero expected value whenever  $i$  misreports unilaterally, the lower expected value between  $\eta_i + c\psi_i$  and  $\eta_i - c\psi_i$  is negative under a sufficiently large  $c$ . Thus, IC can be achieved. Intuitively, with multiple transfer rules, different IC constraints can be satisfied by distinct transfers. Namely, we do not need one transfer rule to satisfy all IC constraints, and thus the full surplus can be extracted under a weaker condition than under Bayesian mechanisms.

To prove the sufficiency direction of Part 2, Lemma A.5 constructs a BB transfer rule (a lottery)  $\psi$  such that (1) each agent's component of  $\psi$  gives him zero interim values on path, and (2) when any agent  $i$  unilaterally deviates from truthful revelation,  $\psi_i$  gives him a non-zero interim value. The common prior assumption is adopted to guarantee BB of  $\psi$ . Then pick a BB transfer rule  $\eta$  to redistribute surplus between agents so that the IR constraint is satisfied for every agent. Given the utility functions and the allocation rule to be implemented, there always exists a sufficiently large constant  $c > 0$  such that the set of ambiguous transfers  $\Phi = \{\eta + c\psi, \eta - c\psi\}$  can implement the allocation rule. The efficiency of the allocation rule does not play a role in the proof of Part 2.<sup>8</sup>

For proof of the sufficiency direction of Part 3, we construct a set of ambiguous transfers  $\Phi = \{\eta + c\psi, \eta - c\psi\}$  that differs slightly from the one for Part 2. The BB lottery  $\psi$  gives every agent zero interim value on path. Since the BDP property holds for  $N - 1$  agents, the lottery parts in  $\Phi$  incentivize truthful reports of  $N - 1$  agents. Let  $\eta$  allocate the total surplus generated by the efficient allocation rule  $q$  to the last agent for whom the BDP property fails. Following the spirit of the VCG mechanism,  $\eta$  aligns the last agent's incentives with the efficiency-maximizing mechanism designer, and thus he will also report truthfully in a private value environment. Unlike Part 2, efficiency of  $q$  plays a role in Part 3.

To this end, we have two remarks on Parts 2 and 3 of Theorem 3.1, when not requiring the mechanism to satisfy the BB condition. First, any ex-post efficient allocation rule under any profile of utility functions is implementable via an IR mechanism with ambiguous transfers if and only if the BDP property holds for all agents. In fact, Bergemann et al. (2012) have designed IR Bayesian mechanisms (which can be viewed as trivial mechanisms with ambigu-

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<sup>8</sup>In fact, by combining our proof with that of Kosenok and Severinov (2008), Part 2 can be extended to implement any ex-ante **socially rational** allocation rule  $q$ , i.e.,  $q$  satisfying  $\sum_{\theta \in \Theta} \sum_{i \in I} u_i(q(\theta), \theta)p(\theta) \geq 0$ .

ous transfers) for implementation under the BDP property, which proves the sufficiency of the remark. The necessity direction of this remark can be proved by the profile of utility functions constructed in Parts 1 and 2 of Theorem 3.1 since the BB condition is not used to derive a contradiction there. Second, any efficient allocation rule under any profile of private value utility functions is implementable via an IR mechanism with ambiguous transfers, regardless of agents' beliefs. This result follows directly from the VCG mechanism, which is a degenerated mechanism with ambiguous transfers. The two remarks on implementation without the BB condition do not rely on the common prior assumption. Since unambiguous IR mechanisms can be constructed, mechanisms with ambiguous transfers do not have an advantage over standard mechanisms in the two new problems. Thus, the main contribution of ambiguous transfers to implementation problems is to guarantee the BB condition without violating IR.

### 3.1 Comparison

Part 1 of Theorem 3.1 is directly comparable to the result of Crémer and McLean (1988). They show that full surplus extraction can be guaranteed via Bayesian mechanisms if and only if the Convex Independence condition, defined below, holds for all agents.

**Definition 3.2:** *The **Convex Independence (CI)** condition holds for agent  $i \in I$  if for any type  $\bar{\theta}_i \in \Theta_i$  and non-negative coefficients  $(c_{\hat{\theta}_i})_{\hat{\theta}_i \in \Theta_i}$ ,  $p_i(\cdot|\bar{\theta}_i) \neq \sum_{\hat{\theta}_i \in \Theta_i \setminus \{\bar{\theta}_i\}} c_{\hat{\theta}_i} p_i(\cdot|\hat{\theta}_i)$ .*

The CI condition fails for  $i$  with positive probability when  $|\Theta_i| > |\Theta_{-i}|$ . For example, when  $|\Theta_2| = 3 > |\Theta_1| = 2$ , the CI condition fails for agent 2 for sure. As another instance, if  $N = 3$  and  $(|\Theta_1|, |\Theta_2|, |\Theta_3|) = (5, 2, 2)$ , it is easy to find a non-negligible set of belief profiles under which agent 1's CI fails. The BDP property is weaker than CI in two aspects. Firstly, the BDP property holds for  $i$  generically even when  $|\Theta_i| > |\Theta_{-i}|$ . Secondly, when  $|\Theta_i| \leq |\Theta_{-i}|$ , the CI condition holds for agent  $i$  generically, but the BDP property further weakens the CI condition by allowing the belief of a type of  $i$  to lie in the convex hull of other beliefs of himself. When the BDP property holds for all agents but CI fails for someone, ambiguous transfers can perform better than Bayesian mechanisms in full surplus extraction.

**Example 3.1:** *This two-agent example demonstrates how ambiguous transfers work and can perform better than Bayesian mechanisms in full surplus extraction.*

*Suppose agent 1 has two types and agent 2 has three. Let agent 1's beliefs satisfy  $(p_1(\theta_2^1|\theta_1^1), p_1(\theta_2^2|\theta_1^1), p_1(\theta_2^3|\theta_1^1)) = (0.2, 0.4, 0.4)$  and  $(p_1(\theta_2^1|\theta_1^2), p_1(\theta_2^2|\theta_1^2), p_1(\theta_2^3|\theta_1^2)) = (0.4, 0.2, 0.4)$ . Agent 2's beliefs are given by  $(p_1(\theta_1^1|\theta_2^1), p_1(\theta_1^2|\theta_2^1)) = (\frac{1}{3}, \frac{2}{3})$ ,  $(p_1(\theta_1^1|\theta_2^2), p_1(\theta_1^2|\theta_2^2)) = (\frac{2}{3}, \frac{1}{3})$ ,*

and  $(p_1(\theta_1^1|\theta_2^3), p_1(\theta_1^2|\theta_2^3)) = (0.5, 0.5)$ . The CI condition fails for agent 2.

In a single unit auction, denote each type- $\theta_i$  agent  $i$ 's private value of winning the good by  $v_i(\theta_i)$ . Suppose  $v_2(\theta_2^1) > v_2(\theta_2^2) > v_2(\theta_2^3) > v_1(\theta_1) > 0$  for all  $\theta_1 \in \Theta_1$ . Crémer and McLean (1988) have shown that full surplus extraction is impossible via a Bayesian mechanism.

Next, we see how ambiguous transfers can help. Let the set of ambiguous transfers be  $\Phi = (\phi^1, \phi^2)$ . Transfers  $\phi^1 = (\phi_1^1, \phi_2^1)$  and  $\phi^2 = (\phi_1^2, \phi_2^2)$  are defined as follows.

$$\phi_i^1(\theta_1, \theta_2) = \begin{cases} c\psi_1(\theta_1, \theta_2), & \text{if } i = 1, \\ -v_2(\theta_2) + c\psi_2(\theta_1, \theta_2), & \text{if } i = 2, \end{cases} \quad \phi_i^2(\theta_1, \theta_2) = \begin{cases} -c\psi_1(\theta_1, \theta_2), & \text{if } i = 1, \\ -v_2(\theta_2) - c\psi_2(\theta_1, \theta_2), & \text{if } i = 2, \end{cases}$$

where  $c \geq 1.5(v_2(\theta_2^1) - v_2(\theta_2^3))$ ,  $\psi_1 : \Theta \rightarrow \mathbb{R}$  is given below, and  $\psi_2 = -\psi_1$ .

$\psi_1(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	-2	-1	2
$\theta_1^2$	1	2	-2

For each type- $\bar{\theta}_i$  agent  $i$ ,  $\psi_i(\bar{\theta}_i, \cdot)$  has zero expected value under belief  $p_i(\cdot|\bar{\theta}_i)$ . When he unilaterally misreports  $\hat{\theta}_i \neq \bar{\theta}_i$ ,  $\psi_i(\hat{\theta}_i, \cdot)$  has non-zero expected value.

Full surplus extraction requires the good to be allocated to agent 2. In addition, agents obtain zero interim payoffs on path under both  $\phi^1$  and  $\phi^2$ . Hence, each IR constraint binds.

When type- $\bar{\theta}_2$  agent 2 misreports  $\hat{\theta}_2 \neq \bar{\theta}_2$ , his worst-case expected payoff is  $v_2(\bar{\theta}_2) - v_2(\hat{\theta}_2) - c|\sum_{\theta_1 \in \Theta_1} \psi_2(\theta_1, \hat{\theta}_2)p_2(\theta_1|\bar{\theta}_2)| < v_2(\bar{\theta}_2) - v_2(\hat{\theta}_2)$ . Therefore, any ‘‘upward’’ misreport of agent 2 results in a negative expected payoff. As  $c \geq 1.5(v_2(\theta_2^1) - v_2(\theta_2^3))$  and  $v_2(\theta_2^1) > v_2(\theta_2^2) > v_2(\theta_2^3)$ , it is easy to verify the three ‘‘downward’’ IC constraints:

$$\begin{aligned} IC(\theta_2^1\theta_2^2) & 0 \geq v_2(\theta_2^1) - v_2(\theta_2^2) - c|\frac{1}{3} \times (-1) + \frac{2}{3} \times 2| = v_2(\theta_2^1) - v_2(\theta_2^2) - c, \\ IC(\theta_2^1\theta_2^3) & 0 \geq v_2(\theta_2^1) - v_2(\theta_2^3) - c|\frac{1}{3} \times 2 + \frac{2}{3} \times (-2)| = v_2(\theta_2^1) - v_2(\theta_2^3) - \frac{2}{3}c, \\ IC(\theta_2^2\theta_2^3) & 0 \geq v_2(\theta_2^2) - v_2(\theta_2^3) - c|\frac{2}{3} \times 2 + \frac{1}{3} \times (-2)| = v_2(\theta_2^2) - v_2(\theta_2^3) - \frac{2}{3}c. \end{aligned}$$

Agent 1's IC constraints can be verified similarly. Hence, full surplus extraction is achieved.

Part 2 of Theorem 3.1 is comparable to the result of Kosenok and Severinov (2008). They prove that when beliefs can be generated by a common prior  $p$ , any efficient allocation rule under any profile of utility functions is implementable via an IR and BB Bayesian mechanism, if and only if the CI condition holds for all  $i \in I$  and the Identifiability condition (defined below) holds for the common prior  $p$ .

**Definition 3.3:** The common prior  $p(\cdot)$  satisfies the **Identifiability** condition if for any prior  $\tilde{p}(\cdot) \in \Delta(\Theta)$  such that  $\tilde{p}(\cdot) \neq p(\cdot)$ , there exists an agent  $i \in I$  and type  $\bar{\theta}_i \in \Theta_i$

with marginal probability  $\tilde{p}(\bar{\theta}_i) > 0$ , such that for any non-negative coefficients  $(c_{\hat{\theta}_i})_{\hat{\theta}_i \in \Theta_i}$ , the conditional probability  $\tilde{p}(\cdot | \bar{\theta}_i) \neq \sum_{\hat{\theta}_i \in \Theta_i} c_{\hat{\theta}_i} p(\cdot | \hat{\theta}_i)$ .

When  $N = 3$  and  $|\Theta_i| \geq 3$  for some  $i \in I$ , or when  $N > 3$ , the Identifiability condition holds for almost all common priors, but the condition fails if the cardinality restriction is not satisfied. In particular, Kosenok and Severinov (2008) have remarked that when  $N = 2$ , only priors consistent with independent beliefs satisfy this condition. Thus, their necessary and sufficient conditions can never hold simultaneously in two-agent settings. In a BB Bayesian mechanism where the common prior does not satisfy the Identifiability condition, some agent  $i$  may have the incentive to misreport in a way that makes the truthful report of some  $j \neq i$  appear untruthful. This is because by BB,  $i$  can benefit from a low expected transfer to  $j$ , which is the punishment due to  $j$ 's (seemingly) untruthful report. However, when the set of ambiguous transfers  $\Phi$  is used,  $i$  does not have such an incentive, because whether misreport of  $j$  would result in a high or low expected transfer to  $j$  remains ambiguous. Hence, with ambiguous transfers, we can relax the Identifiability condition.

The BDP property is weaker than the CI condition, and the Identifiability condition becomes irrelevant in the current environment. When the CI condition fails for some agent or the Identifiability condition fails for the common prior, but the BDP property holds for all agents, ambiguous transfers can perform better than Bayesian mechanisms in implementing efficient allocations via IR and BB mechanisms. In particular, ambiguous transfers can generically resolve the impossibility result on implementing efficient allocation rules via IR and BB Bayesian mechanisms with two agents.

The following example illustrates how ambiguous transfers work.

**Example 3.2:** Consider the same profile of beliefs  $(p_i)_{i \in I}$  as in Example 3.1. The beliefs can be generated by the common prior  $p(\cdot)$  below. Recall the CI condition fails for agent 2. The Identifiability condition also fails for  $p(\cdot)$ . Following Kosenok and Severinov (2008), one can construct utility functions under which an efficient allocation rule  $q$  is not Bayesian implementable. However, we will prove that  $q$  is implementable via ambiguous transfers.

$p(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	0.1	0.2	0.2
$\theta_1^2$	0.2	0.1	0.2

Let the feasible set of alternatives  $A$  be  $\{x_0, x_1, x_2\}$ . The outcome  $x_0$  gives both agents zero payoffs at all type profiles. The payoffs given by  $x_1$  and  $x_2$  are presented in the tables below, where the first component is agent 1's payoff and the second denotes 2's. Assume

$0 < a < B$ .

$u_1(x_1, \theta), u_2(x_1, \theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$	$u_1(x_2, \theta), u_2(x_2, \theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	$a, 0$	$a, a$	$a, a$	$\theta_1^1$	$a, a$	$a - 2B, a + B$	$a, 0$
$\theta_1^2$	$a, 0$	$a, a$	$a, a$	$\theta_1^2$	$a, a$	$a - 2B, a + B$	$a, 0$

The efficient allocation rule is  $q(\theta_1, \theta_2^1) = x_2$  and  $q(\theta_1, \theta_2^2) = q(\theta_1, \theta_2^3) = x_1$  for all  $\theta_1 \in \Theta_1$ . Suppose by contradiction that a BB transfer rule  $\phi = (-\phi_2, \phi_2)$  implements  $q$ . Then

$$\begin{aligned}
IC(\theta_1^1 \theta_1^2) & a - 0.2\phi_2(\theta_1^1, \theta_2^1) - 0.4\phi_2(\theta_1^1, \theta_2^2) - 0.4\phi_2(\theta_1^1, \theta_2^3) \\
& \geq a - 0.2\phi_2(\theta_1^2, \theta_2^1) - 0.4\phi_2(\theta_1^2, \theta_2^2) - 0.4\phi_2(\theta_1^2, \theta_2^3), \\
IC(\theta_1^2 \theta_1^1) & a - 0.4\phi_2(\theta_1^2, \theta_2^1) - 0.2\phi_2(\theta_1^2, \theta_2^2) - 0.4\phi_2(\theta_1^2, \theta_2^3) \\
& \geq a - 0.4\phi_2(\theta_1^1, \theta_2^1) - 0.2\phi_2(\theta_1^1, \theta_2^2) - 0.4\phi_2(\theta_1^1, \theta_2^3), \\
IC(\theta_2^1 \theta_2^2) & a + \frac{1}{3}\phi_2(\theta_1^1, \theta_2^1) + \frac{2}{3}\phi_2(\theta_1^2, \theta_2^1) \geq 0 + \frac{1}{3}\phi_2(\theta_1^1, \theta_2^2) + \frac{2}{3}\phi_2(\theta_1^2, \theta_2^2), \\
IC(\theta_2^2 \theta_2^1) & a + \frac{2}{3}\phi_2(\theta_1^1, \theta_2^2) + \frac{1}{3}\phi_2(\theta_1^2, \theta_2^2) \geq a + B + \frac{2}{3}\phi_2(\theta_1^1, \theta_2^1) + \frac{1}{3}\phi_2(\theta_1^2, \theta_2^1).
\end{aligned}$$

Multiply the inequalities by 0.5, 0.5, 0.3, and 0.3 respectively and sum up. We have  $1.6a \geq 1.3a + 0.3B$ , a contradiction.

For each  $i \in I$  and  $\theta \in \Theta$ , define  $\phi_i^1(\theta) = c\psi_i(\theta)$  and  $\phi_i^2(\theta) = -c\psi_i(\theta)$ , where  $\psi = (\psi_1, \psi_2)$  is defined in Example 3.1 and  $c \geq B$ . Let  $\Phi = \{\phi^1, \phi^2\}$  be ambiguous transfers.

Both  $\phi^1$  and  $\phi^2$  satisfy the BB condition. Each type- $\bar{\theta}_i$  agent  $i$  obtains an interim payoff of  $a > 0$  on path, and thus the IR condition holds. When type- $\theta_2^2$  agent 2 misreports  $\theta_2^1$ , his worst-case expected payoff is  $a + B - c|\frac{2}{3} \times (-2) + \frac{1}{3} \times (1)| = a + B - c \leq a$ . Thus, we have established  $IC(\theta_2^2 \theta_2^1)$ . The other IC constraints can be verified similarly. Therefore, the ambiguous transfers implement  $q$ .

This example can also demonstrate the necessity of the BDP property. Suppose instead that the beliefs satisfy  $\tilde{p}_2(\cdot|\theta_2^1) = \tilde{p}_2(\cdot|\theta_2^2)$  and that an IR and BB mechanism with ambiguous transfers  $(q, \tilde{\Phi})$  implements  $q$ . By adding the following expressions

$$\begin{aligned}
IC(\theta_2^1 \theta_2^2) & \inf_{\tilde{\phi} \in \tilde{\Phi}} \left\{ a + \sum_{\theta_1 \in \Theta_1} \tilde{\phi}_2(\theta_1, \theta_2^1) \tilde{p}_2(\theta_1|\theta_2^1) \right\} \geq \inf_{\tilde{\phi} \in \tilde{\Phi}} \left\{ \sum_{\theta_1 \in \Theta_1} \tilde{\phi}_2(\theta_1, \theta_2^2) \tilde{p}_2(\theta_1|\theta_2^1) \right\}, \\
IC(\theta_2^2 \theta_2^1) & \inf_{\tilde{\phi} \in \tilde{\Phi}} \left\{ a + \sum_{\theta_1 \in \Theta_1} \tilde{\phi}_2(\theta_1, \theta_2^2) \tilde{p}_2(\theta_1|\theta_2^2) \right\} \geq \inf_{\tilde{\phi} \in \tilde{\Phi}} \left\{ a + B + \sum_{\theta_1 \in \Theta_1} \tilde{\phi}_2(\theta_1, \theta_2^1) \tilde{p}_2(\theta_1|\theta_2^2) \right\},
\end{aligned}$$

and taking into account  $\tilde{p}_2(\cdot|\theta_2^1) = \tilde{p}_2(\cdot|\theta_2^2)$ , we have  $2a \geq a + B$ , a contradiction. Hence, implementation via ambiguous transfers cannot be guaranteed without the BDP property.

To show that ambiguous transfers may improve upon Bayesian mechanisms in the context of Part 3 of Theorem 3.1, we present a private value example below.



**Example 3.3:** In this bilateral trade example with private values, the BDP property fails for agent 1 and holds for agent 2. The efficient allocation rule is not implementable via a Bayesian mechanism, but implementable via ambiguous transfers.

Agent 1 is the seller of one indivisible good and agent 2 is the buyer. Each agent has three types. Agents' beliefs  $(p_1, p_2)$  can be generated by a common prior  $p$ . Outcomes in  $A = \{x_0, x_1\}$  are feasible, where  $x_0$  represents no trade and  $x_1$  represents trading. No trade gives both agents zero payoff. Agents' payoffs from outcome  $x_1$ , the efficient allocation rule  $q$ , and the common prior  $p$  are given in the three tables below.

$u_1(x_1, \theta), u_2(x_1, \theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$	$q(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$	$p(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	-19,20	-19,3	-19,1	$\theta_1^1$	$x_1$	$x_0$	$x_0$	$\theta_1^1$	0.1	0.1	0.1
$\theta_1^2$	-2,20	-2,3	-2,1	$\theta_1^2$	$x_1$	$x_1$	$x_0$	$\theta_1^2$	0.1	0.1	0.1
$\theta_1^3$	0,20	0,3	0,1	$\theta_1^3$	$x_1$	$x_1$	$x_1$	$\theta_1^3$	0.2	0.15	0.05

Suppose by way of contradiction that there exists an IR and BB Bayesian mechanism  $\phi = (\phi_1, \phi_2)$  such that  $q$  is implementable. By the BB condition,  $\phi_1 = -\phi_2$ . Similar to Example 3.2, we can express both agents' IR and IC constraints with  $\phi_2$ . Then, by multiplying  $IR(\theta_1^1)$ ,  $IR(\theta_1^3)$ ,  $IC(\theta_1^2\theta_1^1)$ ,  $IR(\theta_2^2)$ ,  $IR(\theta_2^3)$ ,  $IC(\theta_2^1\theta_2^2)$ , and  $IC(\theta_2^3\theta_2^2)$  by 6, 4, 3, 8.75, 1.25, 4, and 1.25 respectively, we obtain that  $57.25 \geq 58.75$ , a contradiction.

To see how ambiguous transfers work, let  $\Phi = \{\phi^1, \phi^2\}$ . Define  $\phi_1^1$  and  $\phi_1^2$  below, which are agent 1's component in  $\phi^1$  and  $\phi^2$  respectively. The constant  $c$  is no less than 12.75.

$\phi_1^1(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	$20 + 9c$	$-18c$	$9c$
$\theta_1^2$	$20 - 3c$	$3 + 6c$	$-3c$
$\theta_1^3$	$20 - 3c$	$3 + 8c$	$1 - 12c$

$\phi_1^2(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	$20 - 9c$	$18c$	$-9c$
$\theta_1^2$	$20 + 3c$	$3 - 6c$	$3c$
$\theta_1^3$	$20 + 3c$	$3 - 8c$	$1 + 12c$

Notice that the BDP property holds for agent 2. It is easy to verify his IR and IC constraints. We first notice that the MEU of agent 2 is always zero on path, which establishes his IR constraints. When type- $\theta_2^1$  agent 2 misreports  $\theta_2^2$  and  $\theta_2^3$  respectively, his MEU becomes  $\frac{51}{4} - c$  and  $\frac{19}{2} - \frac{9}{2}c$  respectively. Similarly, when type- $\theta_2^2$  misreports  $\theta_2^1$  and  $\theta_2^3$ , the MEU is  $-17 - \frac{3}{7}c$  and  $\frac{6}{7} - \frac{24}{7}c$  respectively. At last, when type- $\theta_2^3$  reports  $\theta_2^1$  and  $\theta_2^2$ , his MEU is  $-19 - \frac{9}{5}c$  and  $-\frac{6}{5} - \frac{16}{5}c$ . Hence, when  $c \geq 12.75$ , we can verify agent 2's IC constraints.

As the BDP property fails for agent 1, one may suspect that our mechanism cannot guarantee IC of agent 1. However, this is not the case. We can decompose each of the

potential transfer rule into two parts, a surplus redistribution part  $\eta$  that does not include  $c$  and a lottery part  $\psi$  that is enlarged by the constant  $c$ . Both  $\eta$  and  $\psi$  satisfy the BB condition. Agent 1's components in the potential transfer rules are  $\phi_1^1 = \eta_1 + c\psi_1$  and  $\phi_1^2 = \eta_1 - c\psi_1$ . We cannot rely on the lottery part in ambiguous transfers to incentivize truth-telling of agent 1, since the BDP property fails for agent 1. However, as the surplus redistribution part in  $\phi_1^1$  and  $\phi_1^2$ ,  $\eta_1$ , is a VCG transfer, it guarantees IC of agent 1 under the private value assumption.

$\eta_1(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	20	0	0
$\theta_1^2$	20	3	0
$\theta_1^3$	20	3	1

$\psi_1(\theta)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	9	-18	9
$\theta_1^2$	-3	6	-3
$\theta_1^3$	-3	8	-12

We now formally verify agent 1's IR and IC constraints. When type- $\theta_1^1$  agent 1 truthfully reports  $\theta_1^1$ , his MEU is  $\frac{1}{3}$ . When he misreports  $\theta_1^2$  or  $\theta_1^3$ , his MEU decreases to  $-5$  or  $-11 - \frac{7}{3}c$ . Similarly, when type- $\theta_1^2$  agent 1 truthfully reveals, he obtains MEU of  $\frac{19}{3}$ . But when misreporting  $\theta_1^1$  and  $\theta_1^3$ , his MEU decreases to 6 and  $6 - \frac{7}{3}c$ . By truthfully reporting  $\theta_1^3$ , agent 1's MEU is  $\frac{45}{4}$ . Misreporting  $\theta_1^1$  and  $\theta_1^2$  leads to the lower MEU of  $10 - \frac{9}{8}c$  and  $\frac{89}{8} - \frac{3}{8}c$  respectively. Agent 1's IR and IC constraints are thus established. Indeed, as long as  $c \geq 0$ , his IC constraints are satisfied. This is consistent with the fact that the VCG transfer of agent 1 ensures his IC constraints, rather than the lottery parts in the ambiguous transfers.

Hence,  $q$  is implementable via the IR and BB mechanism with ambiguous transfers.

## 4 Extensions

### 4.1 Relax the common prior assumption

In this section, we study implementation via ambiguous transfers without imposing the assumption that beliefs can be generated by a common prior. We demonstrate with examples that ambiguous transfers may implement Bayesian non-implementable allocation rules.

The common prior assumption is used in Parts 2 and 3 of Theorem 3.1. In fact, without the assumption, the following example shows that the BDP property is no longer sufficient for implementation via an IR and BB mechanism with ambiguous transfers.

**Example 4.1:** Consider an adaptation of Example 3.2 where each agent has two types. In  $A = \{x_0, x_1, x_2\}$ , the payoffs of  $x_1$  and  $x_2$  are presented below. The payoff of  $x_0$  is zero to both agents. Assume  $0 < 16a < B$ .

$u_1(x_1, \theta), u_2(x_1, \theta)$	$\theta_2^1$	$\theta_2^2$
$\theta_1^1$	$a, 0$	$a, a$
$\theta_1^2$	$a, 0$	$a, a$

$u_1(x_2, \theta), u_2(x_2, \theta)$	$\theta_2^1$	$\theta_2^2$
$\theta_1^1$	$a, a$	$a - 2B, a + B$
$\theta_1^2$	$a, a$	$a - 2B, a + B$

The efficient allocation rule is  $q(\theta_1^1, \theta_2^1) = q(\theta_1^2, \theta_2^1) = x_2$  and  $q(\theta_1^1, \theta_2^2) = q(\theta_1^2, \theta_2^2) = x_1$ . Let the beliefs satisfy  $p_1(\theta_2^1|\theta_1^1) = 0.75$ ,  $p_1(\theta_2^1|\theta_1^2) = 0.25$ ,  $p_2(\theta_1^1|\theta_2^1) = 0.7$ , and  $p_2(\theta_1^1|\theta_2^2) = 0.3$ , which cannot be generated by a common prior. The BDP property holds for both agents.

Suppose by contradiction that an IR and BB mechanism with ambiguous transfers  $(q, \Phi)$  implements  $q$ . By IC( $\theta_2^2\theta_1^1$ ), we have

$$\inf_{\tilde{\phi} \in \Phi} \{a + 0.3\tilde{\phi}_2(\theta_1^1, \theta_2^2) + 0.7\tilde{\phi}_2(\theta_1^2, \theta_2^2)\} \geq \inf_{\tilde{\phi} \in \Phi} \{a + B + 0.3\tilde{\phi}_2(\theta_1^1, \theta_2^1) + 0.7\tilde{\phi}_2(\theta_1^2, \theta_2^1)\}.$$

Hence, for all  $\epsilon > 0$ , there exists a BB transfer rule  $\phi = (\phi_1, \phi_2) = (-\phi_2, \phi_2) \in \Phi$  such that

$$a + 0.3\phi_2(\theta_1^1, \theta_2^2) + 0.7\phi_2(\theta_1^2, \theta_2^2) + \epsilon \geq a + B + 0.3\phi_2(\theta_1^1, \theta_2^1) + 0.7\phi_2(\theta_1^2, \theta_2^1). \quad (1)$$

We fix this  $\phi$  for the rest of the example.

Recall from the BB condition, for each  $\tilde{\phi} \in \Phi$ ,  $\tilde{\phi}_1 = -\tilde{\phi}_2$ . Thus, IR( $\theta_1^1$ ) requires

$$\inf_{\tilde{\phi} \in \Phi} \{a - 0.75\tilde{\phi}_2(\theta_1^1, \theta_2^1) - 0.25\tilde{\phi}_2(\theta_1^1, \theta_2^2)\} \geq 0.$$

As a result, the transfer rule  $\phi$  satisfies

$$a - 0.75\phi_2(\theta_1^1, \theta_2^1) - 0.25\phi_2(\theta_1^1, \theta_2^2) \geq 0. \quad (2)$$

Similarly, IR( $\theta_1^2$ ), IR( $\theta_2^1$ ), and IR( $\theta_2^2$ ) imply that

$$IR(\theta_1^2) \quad a - 0.25\phi_2(\theta_1^2, \theta_2^1) - 0.75\phi_2(\theta_1^2, \theta_2^2) \geq 0, \quad (3)$$

$$IR(\theta_2^1) \quad a + 0.7\phi_2(\theta_1^1, \theta_2^1) + 0.3\phi_2(\theta_1^2, \theta_2^1) \geq 0, \quad (4)$$

$$IR(\theta_2^2) \quad a + 0.3\phi_2(\theta_1^1, \theta_2^2) + 0.7\phi_2(\theta_1^2, \theta_2^2) \geq 0. \quad (5)$$

Multiply expressions (1), (2), (3), (4), and (5) by 4, 18, 14, 21, and 11 respectively, add them up, and let  $\epsilon$  go to zero. It follows that  $64a - 4B \geq 0$ , a contradiction with  $0 < 16a < B$ .

Hence,  $q$  is not implementable via an IR and BB mechanism with ambiguous transfers.

A few papers in the literature have documented results related to efficiency maximization without the common prior assumption. Among them, Bergemann et al. (2012) study the implementation problem without the BB condition, Smith (2010) compares the welfare of two public good provision mechanisms, and B6rgers et al. (2015) provide a sufficient condition

under which agents' equilibrium interim payoffs can be arbitrarily increased. The current section provides a general condition under which the first-best efficiency can be guaranteed by an IR and BB mechanism without imposing the common prior assumption, which is new to the literature.

We introduce a notation here. For agents  $i \neq j$  and types  $\theta_i$  and  $\theta_j$ , by slightly abusing notations, we let  $p_j(\theta_i, \cdot | \theta_j)$  be the  $|\Theta_{-i-j}|$ -dimensional vector  $(p_j(\theta_i, \theta_{-i-j} | \theta_j))_{\theta_{-i-j} \in \Theta_{-i-j}}$  when  $N \geq 3$ , and be the number  $p_j(\theta_i | \theta_j)$  when  $N = 2$ .

**Definition 4.1:** *The profile of beliefs  $(p_j)_{j \in I}$  satisfies the **Property I** for agent  $i$  if there do not exist types  $\bar{\theta}_i \neq \hat{\theta}_i$ , a prior  $\mu \in \Delta(\Theta)$ , and constants  $\bar{C} > 0$  and  $\hat{C} > 1$  such that:*

$$\mu(\theta_j) > 0 \text{ and } \mu(\theta_{-j} | \theta_j) = p_j(\theta_{-j} | \theta_j) \text{ for all } (j, \theta_j) \neq (i, \hat{\theta}_i) \text{ and } \theta_{-j} \quad (6)$$

and

$$\hat{C} p_i(\theta_j, \cdot | \hat{\theta}_i) = p_i(\theta_j, \cdot | \bar{\theta}_i) + \bar{C} \frac{p_i(\theta_j | \bar{\theta}_i)}{p_j(\bar{\theta}_i | \theta_j)} p_j(\hat{\theta}_i, \cdot | \theta_j) \text{ for all } j \neq i \text{ and } \theta_j. \quad (7)$$

*The profile of beliefs  $(p_j)_{j \in I}$  satisfies **Property II** for agent  $i$  if there do not exist types  $\bar{\theta}_i \neq \hat{\theta}_i$ , a prior  $\mu \in \Delta(\Theta)$ , and constants  $\bar{C} \geq 1$  and  $\hat{C} > 1$  such that*

$$\mu(\theta_j) > 0 \text{ and } \mu(\theta_{-j} | \theta_j) = p_j(\theta_{-j} | \theta_j) \text{ for all } (j, \theta_j) \neq (i, \hat{\theta}_i) \text{ and } \theta_{-j} \quad (8)$$

and

$$\hat{C} p_i(\theta_j, \cdot | \hat{\theta}_i) = p_i(\theta_j, \cdot | \bar{\theta}_i) + \bar{C} \frac{p_i(\theta_j | \bar{\theta}_i)}{p_j(\bar{\theta}_i | \theta_j)} p_j(\hat{\theta}_i, \cdot | \theta_j) \text{ for all } j \neq i \text{ and } \theta_j. \quad (9)$$

Notice that expressions (6) and (8) are identical, and so are expressions (7) and (9). Hence, in the definitions of Properties I and II, the only difference is the size of  $\bar{C}$ . It is easy to see that when the profile of beliefs  $(p_j)_{j \in I}$  satisfies Property I for agent  $i$ , it also satisfies Property II for agent  $i$ .

From expressions (6) through (9), one can see that whether the profile of beliefs  $(p_j)_{j \in I}$  satisfies Property I (or Property II) for agent  $i$  relies on the entire profile of beliefs  $(p_j)_{j \in I}$ , instead of  $p_i$  alone. Such a feature of Property I/II is different from the BDP property, since whether the BDP property holds for agent  $i$  only depends on  $p_i$ . For convenience, we may also say  $(p_j)_{j \in I}$  satisfies the BDP property for agent  $i$ , but  $p_j$  should be viewed as free variables for any  $j \neq i$  in this statement.

When the profile of beliefs  $(p_j)_{j \in I}$  does not satisfy Property I for agent  $i$ , expressions (6) and (7) need to be satisfied simultaneously. Expression (6) requires that the belief of each type- $\theta_j$  agent  $j$ , except the one of type- $\hat{\theta}_i$  agent  $i$ , can be obtained by updating  $\mu$ . When agents' beliefs have full support and expression (6) holds, expression (7) becomes:

$$\hat{C} \frac{p_i(\theta_{-i} | \hat{\theta}_i)}{p_i(\theta_{-i} | \bar{\theta}_i)} = 1 + \bar{C} \frac{\mu(\hat{\theta}_i, \theta_{-i})}{\mu(\bar{\theta}_i, \theta_{-i})}, \forall \theta_{-i} \in \Theta_{-i}.$$

According to this expression, the ratio of  $i$ 's beliefs under types  $\hat{\theta}_i$  and  $\bar{\theta}_i$  has a linear relationship with the ratio of the prior  $\mu$  at types  $\hat{\theta}_i$  and  $\bar{\theta}_i$ .<sup>9</sup>

We check the two properties for each agent in Example 4.1 below.

**Example 4.1** (continued): *When  $N = 2$ , we can replace expression (7) (and (9)) with*

$$\hat{C} \frac{p_i(\theta_j|\hat{\theta}_i)}{p_i(\theta_j|\bar{\theta}_i)} = 1 + \bar{C} \frac{p_j(\hat{\theta}_i|\theta_j)}{p_j(\bar{\theta}_i|\theta_j)}, \forall \theta_j \in \Theta_j. \quad (10)$$

*For agent  $i = 1$ , we claim that there do not exist types  $\bar{\theta}_1 \neq \hat{\theta}_1$  and constants  $\bar{C} > 0$ ,  $\hat{C} > 1$  such that expression (10) holds. Indeed, if one conjectures that  $\hat{\theta}_1 = \theta_1^1$ , then expression (10) implies  $\hat{C} \left( \frac{0.75}{0.25}, \frac{0.25}{0.75} \right) = (1, 1) + \bar{C} \left( \frac{0.7}{0.3}, \frac{0.3}{0.7} \right)$ . This further requires that  $\bar{C} = -\frac{21}{4}$  and  $\hat{C} = -\frac{15}{4}$ . If we conjecture that  $\hat{\theta}_1 = \theta_1^2$  instead, then according to expression (10),  $\hat{C} \left( \frac{0.25}{0.75}, \frac{0.75}{0.25} \right) = (1, 1) + \bar{C} \left( \frac{0.3}{0.7}, \frac{0.7}{0.3} \right)$ . Again, it follows that  $\bar{C} = -\frac{21}{4}$  and  $\hat{C} = -\frac{15}{4}$ . Neither conjecture leads to  $\bar{C} > 0$  and  $\hat{C} > 1$ . Hence, the profile of beliefs satisfies Property I for agent 1. Since Property II is weaker than I, it follows that the profile of beliefs also satisfies Property II for agent 1.*

*However, the profile of beliefs satisfies neither Property I nor Property II for agent 2. To see this, let  $(i, \bar{\theta}_i, \hat{\theta}_i) = (2, \theta_2^2, \theta_2^1)$ ,  $\mu(\theta_1^1, \theta_2^1) = \frac{27}{64}$ ,  $\mu(\theta_1^1, \theta_2^2) = \frac{9}{64}$ ,  $\mu(\theta_1^2, \theta_2^1) = \frac{7}{64}$ ,  $\mu(\theta_1^2, \theta_2^2) = \frac{21}{64}$ ,  $\bar{C} = \frac{15}{4}$ , and  $\hat{C} = \frac{21}{4}$ . Since expressions (6) through (9) would hold, the profile of beliefs does not satisfy Property I or Property II for agent 2.*

One may ask if it is demanding to require the profile of beliefs to satisfy Properties I and II for agent  $i$ . The answer is no when  $N = 2$  and agent  $j \neq i$  has at least three types, or when  $N \geq 3$ . When  $N = 2$ , recall that expressions (6) and (10) are equivalent. There are  $|\Theta_j|$  linear equations in expression (10). With  $|\Theta_j| > 2$ , it is generically impossible to find  $\bar{C}$  and  $\hat{C}$  satisfying all  $|\Theta_j|$  equations and thus almost all profiles of beliefs  $(p_i, p_j)$  satisfy Properties I and II for agent  $i$ . When  $N \geq 3$ , under almost all profiles of beliefs  $(p_l)_{l \in I}$ , there exist two other agents  $j$  and  $k$ , such that there is no  $\mu \in \Delta(\Theta)$  satisfying  $\mu(\theta_{-j}|\theta_j) = p_j(\theta_{-j}|\theta_j)$  for all  $\theta_j, \theta_{-j}$  and  $\mu(\theta_{-k}|\theta_k) = p_k(\theta_{-k}|\theta_k)$  for all  $\theta_k, \theta_{-k}$ . Hence, almost all profiles of beliefs satisfy Properties I and II for agent  $i$  when  $N \geq 3$ .

To see the connection between the BDP property, Property I, and Property II, we first present the following lemma (proved in Online Appendix B). In the special case that beliefs can be generated by a common prior, the three properties are equivalent.

**Lemma 4.1:** *Suppose the profile of beliefs  $(p_j)_{j \in I}$  can be generated by a common prior  $p$ . For each agent  $i \in I$ , the following three statements are equivalent:*

1. *the BDP property holds for agent  $i$ ;*

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<sup>9</sup>Recall that when  $N \geq 3$ , Assumption 2.1 imposes a weaker requirement than full support beliefs.

2. the profile of beliefs  $(p_j)_{j \in I}$  satisfies Property I for agent  $i$ ;
3. the profile of beliefs  $(p_j)_{j \in I}$  satisfies Property II for agent  $i$ .

However, when we do not impose the common prior assumption, there is no implication relationship between the three properties except that Property I implies Property II. In Online Appendix C, we provide a diagram and examples to support this claim.

In view of Lemma 4.1, the following proposition generalizes Parts 2 and 3 of Theorem 3.1. Since Theorem 3.1 is more elegant and only involves the BDP property in the characterization, we leave it as the main result and Proposition 4.1 as an extension.

**Proposition 4.1:** *Given a profile of beliefs  $(p_k)_{k \in I}$ ,*

1. *if there exists an agent  $i \in I$  for whom the profile of beliefs fails to satisfy Property II or the BDP property, then there exists a profile of utility functions under which an efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers; if the profile of beliefs satisfies both Property I and the BDP property for all agents, then any ex-post efficient allocation rule under any profile of utility functions is implementable via an IR and BB mechanism with ambiguous transfers;*
2. *if there exist two agents  $i \neq j$  such that the profile of beliefs fails to satisfy Property II for  $i$  and fails to satisfy the BDP property for  $j$ , then there exists a profile of **private value** utility functions under which an ex-post efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers; if there do not exist two agents  $i \neq j$  such that the profile of beliefs fails to satisfy Property I for  $i$  and fails to satisfy the BDP property for  $j$ , then any ex-post efficient allocation rule under any profile of **private value** utility functions is implementable via an IR and BB mechanism with ambiguous transfers.*

The proof is relegated to Online Appendix B.

We remark that when the BDP property holds for all agents, the sufficient conditions for implementation in Part 2 of the proposition hold.<sup>10</sup> Hence, when the BDP property holds for all agents, whether beliefs can be generated by a common prior or not, efficient implementation via ambiguous transfers can be guaranteed under private value environments. This result is used in Example 4.3.

In Example 4.2, the sufficient conditions in Part 1 of Proposition 4.1 hold. As a result, any efficient allocation rule is implementable via ambiguous transfers. Since we find an efficient allocation rule that is not implementable via Bayesian mechanisms, we demonstrate that ambiguous transfers may perform better than Bayesian mechanisms.

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<sup>10</sup>This is because there does not exist an agent  $j$  such that the BDP property fails for  $j$ .

**Example 4.2:** Under the following profile of beliefs  $(p_1, p_2)$ , the efficient allocation rule  $q$  is not Bayesian implementable, but it is implementable via ambiguous transfers.

$p_1(\theta_2 \theta_1)$	$\theta_2^1$	$\theta_2^2$	$\theta_2^3$
$\theta_1^1$	$\frac{7}{28}$	$\frac{12}{28}$	$\frac{9}{28}$
$\theta_1^2$	$\frac{13}{28}$	$\frac{12}{28}$	$\frac{3}{28}$

$p_2(\theta_1 \theta_2)$	$\theta_1^1$	$\theta_1^2$	$\theta_1^3$
$\theta_2^1$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
$\theta_2^2$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

The feasible set of outcomes, the payoffs, and the efficient allocation rule are identical to those in Example 3.2, except that  $0 < 8.5a < B$  is imposed. Suppose by contradiction that there exists a BB Bayesian transfer rule  $\phi = (\phi_1, \phi_2) = (-\phi_2, \phi_2) : \Theta \rightarrow \mathbb{R}^2$  implementing  $q$ . As in Example 3.2, by multiplying  $IR(\theta_1^1)$ ,  $IR(\theta_1^2)$ ,  $IR(\theta_2^1)$ ,  $IR(\theta_2^2)$ ,  $IR(\theta_2^3)$ ,  $IC(\theta_1^1\theta_1^2)$ ,  $IC(\theta_1^2\theta_1^1)$ ,  $IC(\theta_2^1\theta_2^2)$ ,  $IC(\theta_2^2\theta_2^1)$ , and  $IC(\theta_2^3\theta_2^2)$  by 7, 7, 3, 8, 3, 3.5, 3.5, 3, 3, 4, and 3 respectively, and summing up, we obtain  $0 \geq 4B - 34a$ , a contradiction. Hence,  $q$  is not Bayesian implementable.

It is easy to see that the BDP property holds for both agents.

We demonstrate below that the profile of beliefs satisfies Property I for agent 1 first. Recall that when  $N = 2$ , expression (7) is equivalent to expression (10). Consider  $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^1, \theta_1^2)$ , there do not exist constants  $\bar{C} > 0$  and  $\hat{C} > 1$  such that  $\hat{C}(\frac{13}{7}, 1, \frac{1}{3}) = (1, 1, 1) + \bar{C}(2, 1, 0.5)$ . A symmetric argument applies to  $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^2, \theta_1^1)$ .

To see the profile of beliefs satisfies Property I for agent 2, notice that for all  $\hat{\theta}_2 \in \Theta_2$ , there never exists  $\mu$  such that expression (6) holds.

By Part 1 of Proposition 4.1,  $q$  is implementable via an IR and BB mechanism with ambiguous transfers. For example, we can consider a BB transfer rule  $\phi^1$ , where agent 1's component  $\phi_1^1(\theta) = 0$  for all  $\theta \in \Theta$ . In the second BB transfer rule  $\phi^2$ , agent 1's component is defined by  $\phi_1^2(\theta_1^1, \theta_1^2) = 4B$ ,  $\phi_1^2(\theta_1^1, \theta_2^2) = -\frac{13}{6}B$ ,  $\phi_1^2(\theta_1^1, \theta_2^3) = 0$ ,  $\phi_1^2(\theta_2^1, \theta_2^2) = -2B$ ,  $\phi_1^2(\theta_2^1, \theta_2^3) = \frac{13}{6}B$ , and  $\phi_1^2(\theta_2^2, \theta_2^3) = 0$ . One can check that the BB mechanism with ambiguous transfers  $(q, \Phi = \{\phi^1, \phi^2\})$  satisfies the conditions of IR and IC.

In the following private value bilateral trading example, there exists an efficient allocation rule  $q$  that is not Bayesian implementable. However, the sufficient conditions in Part 2 of Proposition 4.1 hold, and thus  $q$  is implementable via ambiguous transfers. Hence, ambiguous transfers may perform better than Bayesian mechanisms even when we confine the analysis to private value environments.

**Example 4.3:** Agent 1 is the seller of a unit of indivisible good and 2 is the buyer. Outcomes in  $A = \{x_0, x_1\}$  are feasible. The outcome  $x_0$  represents no trade. The payoffs of  $x_1$ , trading, are given below. The efficient allocation rule is  $q(\theta_1^1, \theta_2^2) = x_0$  and  $q(\theta) = x_1$  for all other  $\theta$ .

$u_1(x_1, \theta), u_2(x_1, \theta)$	$\theta_2^1$	$\theta_2^2$
$\theta_1^1$	-3.5, 4	-3.5, 1
$\theta_1^2$	-0.5, 4	-0.5, 1

Let the beliefs satisfy  $p_1(\theta_2^1|\theta_1^1) = 0.3$ ,  $p_1(\theta_2^1|\theta_1^2) = 0.25$ ,  $p_2(\theta_1^1|\theta_2^1) = 0.3$ , and  $p_2(\theta_1^1|\theta_2^2) = 0.2$ , which cannot be generated by a common prior.

Suppose by way of contradiction that there exists an IR and BB Bayesian transfer rule  $\phi = (-\phi_2, \phi_2) : \Theta \rightarrow \mathbb{R}^2$  that implements  $q$ . As in Example 3.2, we can multiply  $IR(\theta_1^1)$ ,  $IC(\theta_1^2|\theta_1^1)$ ,  $IC(\theta_2^1|\theta_2^2)$ ,  $IR(\theta_2^2)$ , and  $IC(\theta_2^2|\theta_2^1)$  by 10, 8, 4, 10, and 1 respectively and add them up. This gives us  $0 \geq 0.9$ , a contradiction. Therefore,  $q$  is not Bayesian implementable.

However, notice that the BDP property holds for both agents. According to the remark after Proposition 4.1,  $q$  is implementable via an IR and BB mechanism with ambiguous transfers. For example, a mechanism with ambiguous transfer rules  $(q, \Phi = \{\phi^1, \phi^2\})$  can fulfill the goal, where agent 1's component in  $\phi^1$  is defined by  $\phi_1^1(\theta_1^1, \theta_2^1) = 4$ ,  $\phi_1^1(\theta_1^1, \theta_2^2) = 0$ ,  $\phi_1^1(\theta_1^2, \theta_2^1) = 4$ ,  $\phi_1^1(\theta_1^2, \theta_2^2) = 1$ , and his component in  $\phi^2$  is given by  $\phi_1^2(\theta_1^1, \theta_2^1) = -38$ ,  $\phi_1^2(\theta_1^1, \theta_2^2) = 24$ ,  $\phi_1^2(\theta_1^2, \theta_2^1) = 22$ , and  $\phi_1^2(\theta_1^2, \theta_2^2) = -5$ .

## 4.2 Other ambiguity aversion preferences

To check the robustness of our result, we look at alternative preferences of ambiguity aversion in this subsection. One is the  $\alpha$ -maxmin expected utility ( $\alpha$ -MEU) of Ghirardato and Marinacci (2002), and the other is the smooth ambiguity aversion preferences of Klibanoff et al. (2005). When agents have these preferences, the mechanism designer may still benefit from introducing ambiguous transfers.

Ghirardato and Marinacci (2002) introduce the  $\alpha$ -MEU, which is a generalization of the MEU. Under an environment described in Section 2, a type- $\theta_i$  agent  $i$  with  **$\alpha$ -maxmin expected utility** has the following interim utility level from participating and reporting truthfully when  $\Phi$  is the set of ambiguous transfers:

$$\alpha \inf_{\phi \in \Phi} \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) p_i(\theta_{-i}|\theta_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) \right\} \\ + (1 - \alpha) \sup_{\phi \in \Phi} \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) p_i(\theta_{-i}|\theta_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) \right\},$$

where  $\alpha \in [0, 1]$ . An agent is said to be ambiguity-averse if  $\alpha > 0.5$ . The MEU preferences adopted in earlier sections correspond to the case  $\alpha = 1$ .



Alternatively, an agent  $i$  with **smooth ambiguity aversion** has a utility function of

$$\int_{\pi \in \Delta(\Phi)} v \left( \int_{\phi \in \Phi} \left( \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \phi_i(\theta_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i) \right) d\pi \right) d\mu,$$

where for each distribution  $\pi \in \Delta(\Phi)$ ,  $\pi(\phi)$  measures the subjective density that  $\phi$  is the true transfer rule chosen by the mechanism designer; for each distribution  $\mu \in \Delta(\Delta(\Phi))$ ,  $\mu(\pi)$  measures the subjective density that  $\pi \in \Delta(\Phi)$  is the right density function the mechanism designer uses to choose the transfer rule;  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function that characterizes ambiguity attitude, where a strictly concave  $v$  implies ambiguity aversion.

Under the  $\alpha$ -MEU preferences with  $\alpha > 0.5$  or the smooth ambiguity aversion preferences with a strictly concave  $v$ , the sufficiency part of Theorem 3.1 still holds. We can construct ambiguous transfers in the same way as those under the MEU preferences except for choosing a potentially different multiplier  $c$ .

As an illustration, we demonstrate with Example 3.2. Let  $v$  be a strictly increasing and strictly concave function. Consider the same transfers as  $\phi^1$  and  $\phi^2$  except for a potentially different multiplier  $c$ . Then it is easy to verify individual rationality and budget balance. Each  $\pi \in \Delta(\Phi)$  is a Bernoulli distribution between  $\phi^1$  and  $\phi^2$ . Let  $\mu$  be the uniform distribution over  $\Delta(\Phi)$  for example. As an illustration, we check  $IC(\theta_2^2 \theta_2^1)$ . Truth-telling always gives agent 2 an expected utility of

$$\int_0^1 v(\mu a + (1 - \mu)a) d\mu = v(a).$$

By misreporting from  $\theta_2^2$  to  $\theta_2^1$ , agent 2 gets an interim utility of

$$\int_0^1 v(\mu(a + B + c) + (1 - \mu)(a + B - c)) d\mu.$$

For  $v$  sufficiently concave or  $c$  sufficiently large, the above expression has a value no more than  $v(a)$ , implying that  $IC(\theta_2^2 \theta_2^1)$  holds. One can verify other IC constraints as well.

## 5 Conclusion

This paper introduces ambiguous transfers to study full surplus extraction and implementation of an efficient allocation rule via an individually rational and budget-balanced mechanism. We show that the BDP property is necessary and sufficient in both problems, which is weaker than the necessary and sufficient condition for full surplus extraction and implementation via Bayesian mechanisms. Hence, ambiguous transfers can go beyond Bayesian mechanisms. In particular, under two-agent settings, ambiguous transfers offer a resolution to overcome the negative results on two-agent IR and BB implementation problems generically.

## A Appendix

We introduce a few notations. For any positive integer  $K$ , the vector  $\mathbf{0} \in \mathbb{R}^K$  is a vector of  $K$  zeros. Let  $\mathbb{R}_+^K$  denote  $\{v \in \mathbb{R}^K | v_k \geq 0, \forall k = 1, \dots, K\}$ , i.e., the set of all non-negative vectors of dimension  $K$ . The set of all non-negative and non-zero vectors of dimension  $K$  is denoted by  $\mathbb{R}_+^K \setminus \{\mathbf{0}\}$ .

**Lemma A.1:** *Given a profile of beliefs  $(p_j)_{j \in I}$ , if the BDP property holds for an agent  $i \in I$ , then for all  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$ , there exists a transfer rule  $\psi^{\bar{\theta}_i, \hat{\theta}_i} : \Theta \rightarrow \mathbb{R}^N$  such that*

1.  $\sum_{\theta_{-j} \in \Theta_{-j}} \psi_j^{\bar{\theta}_i, \hat{\theta}_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j} | \theta_j) = 0$  for all  $j \in I$  and  $\theta_j \in \Theta_j$ ;
2.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i^{\bar{\theta}_i, \hat{\theta}_i}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) < 0$ .

*Proof.* We first define vectors  $p_{\theta_j, \theta'_j}$  for all  $j \in I$  and  $\theta_j, \theta'_j \in \Theta_j$ . Each  $p_{\theta_j, \theta'_j}$  has  $N \times |\Theta|$  dimensions and every dimension corresponds to an agent and a type profile. For any  $j \in I$  and  $\theta_j, \theta'_j \in \Theta_j$ , whenever there exists  $\theta_{-j} \in \Theta_{-j}$  such that a dimension of  $p_{\theta_j, \theta'_j}$  corresponds to agent  $j$  and type profile  $(\theta'_j, \theta_{-j})$ , let this dimension be  $p_j(\theta_{-j} | \theta_j)$ . Thus, we have defined  $|\Theta_{-j}|$  dimensions of the vector  $p_{\theta_j, \theta'_j}$ . Let all other dimensions of  $p_{\theta_j, \theta'_j}$  be 0.<sup>11</sup>

Suppose by way of contradiction that the BDP property holds for agent  $i$ , but there exist different types  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ , such that no  $\psi^{\bar{\theta}_i, \hat{\theta}_i}$  satisfies the two conditions stated in Lemma A.1. By Fredholm's theorem of the alternative, there exist coefficients  $(a_{\theta_j})_{j \in I, \theta_j \in \Theta_j}$  such that

$$p_{\bar{\theta}_i, \hat{\theta}_i} = \sum_{j \in I} \sum_{\theta_j \in \Theta_j} a_{\theta_j} p_{\theta_j, \theta_j}.$$

By focusing on each dimension of  $p_{\bar{\theta}_i, \hat{\theta}_i}$  that corresponds to agent  $i$  and type profile  $(\hat{\theta}_i, \theta_{-i})$ , we know that  $p_i(\theta_{-i} | \bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i} | \hat{\theta}_i)$  for all  $\theta_{-i} \in \Theta_{-i}$ . Adding this expression over  $\theta_{-i} \in \Theta_{-i}$  yields  $a_{\hat{\theta}_i} = 1$ . Hence,  $p_i(\cdot | \bar{\theta}_i) = p_i(\cdot | \hat{\theta}_i)$ , contradicting the BDP property.  $\square$

**Lemma A.2:** *For any  $K \times K$  matrix  $X = (x_{k\bar{k}})$  whose diagonal elements are all negative, there exists a vector  $\lambda \in \mathbb{R}_+^K \setminus \{\mathbf{0}\}$  such that  $\sum_{\bar{k}=1}^K x_{k\bar{k}} \lambda_{\bar{k}} \neq 0$  for all  $k = 1, \dots, K$ .*

*Proof.* We prove the result by induction.

When  $K = 1$ . Pick an arbitrary  $\lambda_1 > 0$ . As  $x_{11} < 0$ , the statement holds for 1.

Suppose the statement holds for  $K - 1$ , where  $K \geq 2$ . Consider any  $K \times K$  matrix  $X$  with negative diagonal elements. By the supposition for the northwest  $K - 1$  by  $K - 1$  block,

<sup>11</sup>As an illustration, consider  $I = \{1, 2\}$  and  $\Theta = \{(\theta_1^1, \theta_2^1), (\theta_1^1, \theta_2^2), (\theta_1^2, \theta_2^1), (\theta_1^2, \theta_2^2)\}$ . For each  $p_{\theta_j, \theta'_j}$ , its first (last) four dimensions correspond to agent 1 (agent 2) and the type profile  $(\theta_1^1, \theta_2^1), (\theta_1^1, \theta_2^2), (\theta_1^2, \theta_2^1)$ , and  $(\theta_1^2, \theta_2^2)$  respectively. Then for example,  $p_{\theta_2^2, \theta_1^1} = (0, 0, 0, 0, p_2(\theta_1^1 | \theta_2^2), 0, p_2(\theta_1^2 | \theta_2^2), 0)$ .

there exists a non-zero vector  $(\lambda_1, \dots, \lambda_{K-1}) \in \mathbb{R}_+^{K-1} \setminus \{\mathbf{0}\}$  such that  $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$  for all  $k = 1, \dots, K-1$ .

*Case 1.* Suppose  $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda_{\tilde{k}} \neq 0$ . Let  $\lambda_K = 0$ , and thus the statement holds for  $K$ .

*Case 2.* Suppose  $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda_{\tilde{k}} = 0$  and there exists  $k_0 \in \{1, \dots, K-1\}$  such that  $x_{Kk_0} \lambda_{k_0} \neq 0$ . Let  $(\lambda'_1, \dots, \lambda'_{K-1}) = (\lambda_1, \dots, \lambda_{k_0-1}, \lambda_{k_0} + \epsilon, \lambda_{k_0+1}, \dots, \lambda_{K-1})$  for  $\epsilon > 0$ . Then  $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda'_{\tilde{k}} \neq 0$ . When  $\epsilon$  is sufficiently close to zero, as  $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$  for all  $k = 1, \dots, K-1$ , we also have  $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda'_{\tilde{k}} \neq 0$  for all  $k = 1, \dots, K-1$ . Thus, we can replace  $(\lambda_1, \dots, \lambda_{K-1})$  with  $(\lambda'_1, \dots, \lambda'_{K-1})$  and go back to Case 1.

*Case 3.* Suppose  $x_{K\tilde{k}} \lambda_{\tilde{k}} = 0$  for all  $\tilde{k} = 1, \dots, K-1$ . Pick any  $\lambda_K > 0$  such that  $\lambda_K \neq -\frac{\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}}}{x_{kK}}$  for all  $k = 1, \dots, K-1$  with  $x_{kK} \neq 0$ . The statement thus holds for  $K$ .  $\square$

**Lemma A.3:** *Given a profile of beliefs  $(p_i)_{i \in I}$ , if the BDP property holds for all agents, then there exists a transfer rule  $\psi : \Theta \rightarrow \mathbb{R}^N$  such that*

1.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i) = 0$  for all  $i \in I$  and  $\theta_i \in \Theta_i$ ;
2.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) \neq 0$  for all  $i \in I$  and  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$ .

*Proof.* Let  $K$  be the cardinality of the set  $\mathcal{K} = \{(\bar{\theta}_i, \hat{\theta}_i) | i \in I, \bar{\theta}_i, \hat{\theta}_i \in \Theta_i, \bar{\theta}_i \neq \hat{\theta}_i\}$ . Let  $f : \mathcal{K} \rightarrow \{1, \dots, K\}$  be a one-to-one mapping, which allows us to index the elements of  $\mathcal{K}$ .

For all  $k, \tilde{k} \in \{1, \dots, K\}$  ( $k, \tilde{k}$  may be equal), where  $f^{-1}(k) = (\bar{\theta}_i, \hat{\theta}_i)$  and  $f^{-1}(\tilde{k}) = (\bar{\theta}_j, \hat{\theta}_j)$ , define a number  $x_{k\tilde{k}} = \sum_{\theta_{-i} \in \Theta_{-i}} \psi_i^{\bar{\theta}_j, \hat{\theta}_j}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i)$ , where each transfer rule  $\psi_i^{\bar{\theta}_j, \hat{\theta}_j}$  is defined and proved to exist in Lemma A.1. Recall the second condition of  $\psi_i^{\bar{\theta}_j, \hat{\theta}_j}$  implies that  $x_{k\tilde{k}} < 0$  for all  $\tilde{k} = 1, \dots, K$ . Then,  $X \equiv (x_{k\tilde{k}})$  is a  $K \times K$  matrix with negative diagonal elements. By Lemma A.2, there exists  $\lambda \in \mathbb{R}_+^K \setminus \{\mathbf{0}\}$  such that  $\sum_{\tilde{k}=1}^K x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$  for all  $k = 1, \dots, K$ . Hence, for all  $(\bar{\theta}_i, \hat{\theta}_i) \in \mathcal{K}$ ,

$$\sum_{\tilde{k}=1}^K \left[ \sum_{\theta_{-i} \in \Theta_{-i}} \psi_i^{f^{-1}(\tilde{k})}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) \right] \lambda_{\tilde{k}} = \sum_{\theta_{-i} \in \Theta_{-i}} \left[ \sum_{\tilde{k}=1}^K \lambda_{\tilde{k}} \psi_i^{f^{-1}(\tilde{k})}(\hat{\theta}_i, \theta_{-i}) \right] p_i(\theta_{-i} | \bar{\theta}_i) \neq 0. \quad (11)$$

Define a new transfer rule  $\psi$  by making a linear combination of the rules  $(\psi^{f^{-1}(\tilde{k})})_{\tilde{k}=1, \dots, K}$  such that  $\psi \equiv \sum_{\tilde{k}=1}^K \lambda_{\tilde{k}} \psi^{f^{-1}(\tilde{k})}$ . Thus, by expression (11), the transfer rule  $\psi$  satisfies the second condition of this lemma. The first condition of this lemma also holds for  $\psi$  because each  $\psi^{f^{-1}(\tilde{k})}$  satisfies this condition.  $\square$

**Lemma A.4:** *Given a profile of beliefs that can be generated by a common prior  $p$ , if the BDP property holds for agent  $i$ , then for all  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$ , there exists a transfer rule  $\psi^{\bar{\theta}_i, \hat{\theta}_i} : \Theta \rightarrow \mathbb{R}^N$  such that,*

1.  $\sum_{j \in I} \psi_j^{\bar{\theta}_i, \hat{\theta}_i}(\theta) = 0$  for all  $\theta \in \Theta$ ;
2.  $\sum_{\theta_{-j} \in \Theta_{-j}} \psi_j^{\bar{\theta}_i, \hat{\theta}_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j} | \theta_j) = 0$  for all  $j \in I$  and  $\theta_j \in \Theta_j$ ;
3.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i^{\bar{\theta}_i, \hat{\theta}_i}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) < 0$ .

*Proof.* For each  $\theta \in \Theta$ , define a  $N \times |\Theta|$ -dimensional vector  $e_\theta$  below. Every dimension corresponds to an agent and a type profile. Let all dimensions of  $e_\theta$  that correspond to an agent and the type profile  $\theta$  be 1. Let all other dimensions of  $e_\theta$  be 0.<sup>12</sup> Vectors  $(p_{\theta_j, \theta'_j})_{j \in I, \theta_j, \theta'_j \in \Theta_j}$  have been defined in Lemma A.1.

Suppose by way of contradiction that the BDP property holds for agent  $i$ , but there exist types  $\bar{\theta}_i \neq \hat{\theta}_i$ , such that no transfer rule  $\psi^{\bar{\theta}_i, \hat{\theta}_i}$  satisfies the three conditions. By Fredholm's theorem of the alternative, there exist coefficients  $(a_{\theta_j})_{j \in I, \theta_j \in \Theta_j}$  and  $(b_\theta)_{\theta \in \Theta}$  such that

$$p_{\bar{\theta}_i, \hat{\theta}_i} = \sum_{j \in I} \sum_{\theta_j \in \Theta_j} a_{\theta_j} p_{\theta_j, \theta_j} + \sum_{\theta \in \Theta} b_\theta e_\theta. \quad (12)$$

Fix any agent  $j \neq i$ . All elements of  $p_{\bar{\theta}_i, \hat{\theta}_i}$  corresponding to agent  $j$  are zero. All those corresponding to agent  $i$  and  $\bar{\theta}_i$  are zero. Those corresponding to agent  $i$  and  $\hat{\theta}_i$  may not be zero. The three observations, along with expression (12), imply that

$$0 = a_{\theta_j} p_j(\theta_{-j} | \theta_j) + b_\theta, \forall \theta \in \Theta, \quad (13)$$

$$0 = a_{\bar{\theta}_i} p_i(\theta_{-i} | \bar{\theta}_i) + b_{\bar{\theta}_i, \theta_{-i}}, \forall \theta_{-i} \in \Theta_{-i}, \quad (14)$$

$$p_i(\theta_{-i} | \bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i} | \hat{\theta}_i) + b_{\hat{\theta}_i, \theta_{-i}}, \forall \theta_{-i} \in \Theta_{-i}. \quad (15)$$

Choosing  $\theta_i = \bar{\theta}_i$  in expression (13) and cancelling  $b_{\bar{\theta}_i, \theta_{-i}}$  in expressions (13) and (14) yield  $a_{\theta_j} p_j(\bar{\theta}_i, \theta_{-i-j} | \theta_j) = a_{\bar{\theta}_i} p_i(\theta_{-i} | \bar{\theta}_i)$  for all  $\theta_{-i}$ .<sup>13</sup> Since beliefs can be generated by a common prior  $p$ , we further have that  $a_{\theta_j} \frac{p(\bar{\theta}_i, \theta_{-i})}{p(\theta_j)} = a_{\bar{\theta}_i} \frac{p(\bar{\theta}_i, \theta_{-i})}{p(\bar{\theta}_i)}$  for all  $\theta_{-i} \in \Theta_{-i}$ . Summing across all  $\theta_{-i} \in \Theta_{-i}$  yields  $a_{\theta_j} = a_{\bar{\theta}_i} \frac{p(\theta_j)}{p(\bar{\theta}_i)}$  for all  $\theta_j \in \Theta_j$ .

By choosing  $\theta_i = \hat{\theta}_i$  in expression (13) and plugging in  $a_{\theta_j}$  derived in the previous paragraph, we know  $b_{\hat{\theta}_i, \theta_{-i}} = -a_{\bar{\theta}_i} \frac{p(\theta_j)}{p(\bar{\theta}_i)} p_j(\hat{\theta}_i, \theta_{-i-j} | \theta_j) = -a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)} p_i(\theta_{-i} | \hat{\theta}_i)$  for all  $\theta_{-i}$ .

Plugging  $b_{\hat{\theta}_i, \theta_{-i}}$  above into expression (15) yields  $p_i(\theta_{-i} | \bar{\theta}_i) = (a_{\hat{\theta}_i} - a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)}) p_i(\theta_{-i} | \hat{\theta}_i)$  for all  $\theta_{-i}$ . Hence,  $a_{\hat{\theta}_i} - a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)} = 1$  and  $p_i(\cdot | \bar{\theta}_i) = p_i(\cdot | \hat{\theta}_i)$ , a contradiction.  $\square$

**Lemma A.5:** *Given a profile of beliefs that can be generated by a common prior  $p$ , if the BDP property holds for all agents, then there exists a transfer rule  $\psi : \Theta \rightarrow \mathbb{R}^N$  such that*

<sup>12</sup>As an illustration, recall the same example as in footnote <sup>11</sup>. One has  $e_{(\theta_1^2, \theta_2^2)} = (0, 0, 1, 0, 0, 0, 1, 0)$ .

<sup>13</sup>When  $N = 2$ , this expression abuses notations slightly as the left-hand side should be  $a_{\theta_j} p_j(\bar{\theta}_i | \theta_j)$ .

1.  $\sum_{i \in I} \psi_i(\theta) = 0$  for all  $\theta \in \Theta$ ;
2.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i) = 0$  for all  $i \in I$  and  $\theta_i \in \Theta_i$ ;
3.  $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \hat{\theta}_i) \neq 0$  for all  $i \in I$  and  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$ .

*Proof.* One can construct a linear combination of transfer rules developed in Lemma A.4 such that the combination satisfies the three conditions here. The detailed argument is omitted as it is analogous to Lemma A.3.  $\square$

**Proof of Theorem 3.1. Necessity of Parts 1 and 2.** Suppose there exists  $i \in I$  and  $\bar{\theta}_i \neq \hat{\theta}_i$  such that  $p_i(\cdot | \bar{\theta}_i) = p_i(\cdot | \hat{\theta}_i)$ . Consider an adaptation of the utility functions constructed by Kosenok and Severinov (2008). Let the set of feasible outcomes be  $A = \{x_0, x_1, x_2\}$ , where agents' payoffs of consuming  $x_0$  are zero. The payoffs for agent  $i$  and all  $j \neq i$  to consume  $x_1$  and  $x_2$  are given below with  $0 < a < B$ . Note that for each agent in the environment, his payoff from  $x_1$  and  $x_2$  is constant for all  $\theta_{-i} \in \Theta_{-i}$ .

	$u_i(x_1, (\theta_i, \theta_{-i}))$	$u_j(x_1, (\theta_i, \theta_{-i}))$	$u_i(x_2, (\theta_i, \theta_{-i}))$	$u_j(x_2, (\theta_i, \theta_{-i}))$
$\theta_i = \bar{\theta}_i$	a	a	a+B	a-2B
$\theta_i = \hat{\theta}_i$	0	a	a	a
$\theta_i \neq \bar{\theta}_i, \hat{\theta}_i$	a	a	0	a

The efficient allocation rule is  $q(\theta) = x_2$  if  $\theta_i = \hat{\theta}_i$  and  $q(\theta) = x_1$  elsewhere.

Suppose by way of contradiction that full surplus extraction can be achieved by a mechanism with ambiguous transfers  $(q, \Phi)$ . By  $IC(\bar{\theta}_i, \hat{\theta}_i)$  and  $IC(\hat{\theta}_i, \bar{\theta}_i)$ ,

$$\inf_{\phi \in \Phi} \left\{ a + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\bar{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) \right\} \geq \inf_{\phi \in \Phi} \left\{ a + B + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) \right\},$$

$$\inf_{\phi \in \Phi} \left\{ a + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \hat{\theta}_i) \right\} \geq \inf_{\phi \in \Phi} \left\{ 0 + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\bar{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \hat{\theta}_i) \right\}.$$

Recall that  $p_i(\cdot | \bar{\theta}_i) = p_i(\cdot | \hat{\theta}_i)$ . Adding these two inequalities gives  $2a \geq a + B$ , a contradiction. Therefore, the condition that the BDP property holds for all agents is necessary to guarantee full surplus extraction via a mechanism with ambiguous transfers.

To prove that the same condition is necessary for IR and BB implementation via a mechanism with ambiguous transfers, we can adopt the same argument.

**Necessity of Part 3.** By relabeling the indices, we assume without loss of generality that agent 1 has identical beliefs under  $\theta_1^1$  and  $\theta_1^2$ , that agent 2 has identical beliefs under

$\theta_2^1$  and  $\theta_2^2$ , and that  $|\Theta_2| \geq |\Theta_1|$ . For each agent  $i$ , let  $\theta_i$  and  $\theta_{-i}$  be generic elements of  $\Theta_i$  and  $\Theta_{-i}$ . For convenience,  $\theta_1^m$  and  $\theta_2^n$  are also used to represent generic elements of  $\Theta_1$  and  $\Theta_2$ . To avoid introducing additional notations, we ignore any notation  $\theta_{-1-2}$  if  $N = 2$ . Now we construct a profile of private value utility functions such that an efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers. This would establish the necessity of the condition that at least  $N - 1$  agents satisfy the BDP property.

Let agent 1 own a unit of private good and all others be potential buyers. The set of feasible outcomes is  $A = \{x_0, x_1^2, \dots, x_1^n\}$ . The outcome  $x_0$  means no trade. For each  $i \neq 1$ ,  $x_1^i$  means that trade occurs and that agent  $i$  receives the good.

For each buyer  $i \neq 1$ ,  $v_i(\theta_i)$  represents agent  $i$ 's private value of receiving the good, i.e.,  $u_i(x_1^i, \theta_i) = v_i(\theta_i)$ . When trading, type- $\theta_1$  agent 1 (seller) incurs a cost of production  $v_1(\theta_1) < 0$ . Thus,  $u_1(x_1^i, \theta_1) = v_1(\theta_1)$ . For any other potential buyer  $j \neq i$  and  $j \neq 1$  who does not receive the good from such a trade, agent  $j$ 's utility is zero, i.e.,  $u_j(x_1^i, \theta_j) = 0$ . The outside option, no trade, gives all agents zero utility. Hence,  $u_i(x_0, \theta_i) = 0$  for all  $i \in I$ .

Suppose the private value of trading follows the ranking  $v_2(\theta_2^1) > -v_1(\theta_1^1) > v_2(\theta_2^2) > -v_1(\theta_1^2) > \dots > v_2(\theta_2^{|\Theta_1|}) > -v_1(\theta_1^{|\Theta_1|}) > 0$ . Furthermore, let  $-v_1(\theta_1^{|\Theta_1|}) > v_i(\theta_i) > 0$  for all  $v_i(\theta_i)$  that haven't been ranked.

According to the ranking above, the efficient allocation rule is given by  $q(\theta) = x_1^2$  if  $v_1(\theta_1^m) + v_2(\theta_2^n) > 0$ , and  $q(\theta) = x_0$  otherwise. Thus, agent 1 should either sell the good to agent 2 or not sell it to anyone. Note that  $v_1(\theta_1^m) + v_2(\theta_2^n) \neq 0$  by construction.

Suppose by way of contradiction that an IR and BB mechanism with ambiguous transfers, denoted by  $\mathcal{M} = (q, \Phi)$ , implements  $q$ . By the IR condition, for each  $i \in I$  and  $\theta_i$ , type- $\theta_i$  agent  $i$ 's MEU from participation is  $U_{\theta_i} \geq 0$ . Hence,

$$\inf_{\phi \in \Phi} \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\theta_i, \theta_{-i}), \theta_i) p_i(\theta_{-i} | \theta_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i) \right\} = U_{\theta_i} \geq 0.$$

This further implies that for each  $\phi \in \Phi$ ,  $i \in I$ , and  $\theta_i \in \Theta_i$ ,

$$\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i) \geq U_{\theta_i} - \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\theta_i, \theta_{-i}), \theta_i) p_i(\theta_{-i} | \theta_i). \quad (16)$$

We fix any  $\phi \in \Phi$  now. For each  $i$  and  $\theta_i$ , multiply the above inequality by  $p(\theta_i)$ . Then sum across all  $i$  and  $\theta_i$ . By the common prior assumption and the BB condition, the left-hand side of the aggregated inequality is zero. Hence, the aggregated expression is

$$0 \geq \sum_{i \in I} \sum_{\theta_i \in \Theta_i} p(\theta_i) U_{\theta_i} + \sum_m p(\theta_1^m) (-v_1(\theta_1^m) \sum_{n \leq m} p(\theta_2^n | \theta_1^m)) + \sum_n p(\theta_2^n) (-v_2(\theta_2^n) \sum_{m \geq n} p(\theta_1^m | \theta_2^n)). \quad (17)$$

From  $IC(\theta_1^2\theta_1^1)$ , we know

$$U_1^2 \geq \inf_{\phi \in \Phi} \{v_1(\theta_1^2)p(\theta_2^1|\theta_1^2) + \sum_{\theta_{-1}} \phi_1(\theta_1^1, \theta_{-1})p(\theta_{-1}|\theta_1^2)\}.$$

Thus, for each  $\epsilon > 0$ , there exists  $\phi^1 \in \Phi$  satisfying

$$-\sum_{\theta_{-1}} \phi_1^1(\theta_1^1, \theta_{-1})p_1(\theta_{-1}|\theta_1^2) + \epsilon \geq -U_1^2 + v_1(\theta_1^2)p(\theta_2^1|\theta_1^2).$$

Notice that  $p_1(\cdot|\theta_1^1) = p_1(\cdot|\theta_1^2)$ . Add the above expression and (16), where  $i = 1$ ,  $\theta_i = \theta_1^1$ , and  $\phi = \phi^1$ . Then, let  $\epsilon$  go to zero. We obtain that

$$U_{\theta_1^2} \geq U_{\theta_1^1} + (v_1(\theta_1^2) - v_1(\theta_1^1))p(\theta_2^1|\theta_1^1). \quad (18)$$

Similarly, from  $IC(\theta_2^1\theta_2^2)$ , expression (16) with  $i = 2$  and  $\theta_i = \theta_2^2$ , and  $p_2(\cdot|\theta_2^1) = p_2(\cdot|\theta_2^2)$ , we can obtain

$$U_{\theta_2^1} \geq U_{\theta_2^2} + (v_2(\theta_2^1) - v_2(\theta_2^2)) \sum_{m \geq 2} p(\theta_1^m|\theta_2^1). \quad (19)$$

By plugging the above two inequalities into expression (17), we know

$$\begin{aligned} 0 \geq & \sum_m p(\theta_1^m) (-v_1(\theta_1^m) \sum_{n \leq m} p(\theta_2^n|\theta_1^m)) + p(\theta_1^2)(v_1(\theta_1^2) - v_1(\theta_1^1))p(\theta_2^1|\theta_1^1) \\ & + \sum_n p(\theta_2^n) (-v_2(\theta_2^n) \sum_{m \geq n} p(\theta_1^m|\theta_2^n)) + p(\theta_2^1)(v_2(\theta_2^1) - v_2(\theta_2^2)) \sum_{m \geq 2} p(\theta_1^m|\theta_2^1). \end{aligned} \quad (20)$$

In the right-hand side of the above expression, the coefficients of  $v_1(\theta_1^1)$  and  $v_2(\theta_2^1)$  are

$$\begin{aligned} & -p(\theta_1^1)p(\theta_2^1|\theta_1^1) - p(\theta_1^2)p(\theta_2^1|\theta_1^1) < -p(\theta_1^1)p(\theta_2^1|\theta_1^1) = -p(\theta_2^1)p(\theta_1^1|\theta_2^1), \\ & -p(\theta_2^1) + p(\theta_2^1) \sum_{m \geq 2} p(\theta_1^m|\theta_2^1) = -p(\theta_2^1)p(\theta_1^1|\theta_2^1), \end{aligned}$$

respectively, where the strict inequality follows from Assumption 2.1 and the equality in the first expression follows from the common prior assumption. Hence, if we let  $v_1(\theta_1^1)$  and  $v_2(\theta_2^1)$  be sufficiently close in absolute value and all other values  $v_i(\theta_i) \neq v_1(\theta_1^1), v_2(\theta_2^1)$  be sufficiently close to zero, then the right-hand side of expression (20) is positive, a contradiction.

Therefore,  $q$  cannot be implemented via an IR and BB mechanism with ambiguous transfers.

**Sufficiency of Part 1.** Pick an arbitrary ex-post efficient allocation rule  $q$ . Define two transfer rules  $\phi$  and  $\phi'$  by  $\phi_i = \eta_i + c\psi_i$  and  $\phi'_i = \eta_i - c\psi_i$  for all  $i \in I$ , where  $\psi$  is defined in and proved to exist by Lemma A.3,  $\eta_i(\theta) = -u_i(q(\theta), \theta)$  for all  $\theta \in \Theta$ , and  $c$  is no less than

$$\max_{\substack{i \in I, \hat{\theta}_i, \bar{\theta}_i \in \Theta_i, \\ \hat{\theta}_i \neq \bar{\theta}_i}} \frac{\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), (\bar{\theta}_i, \theta_{-i})) - u_i(q(\hat{\theta}_i, \theta_{-i}), (\hat{\theta}_i, \theta_{-i}))] p_i(\theta_{-i}|\bar{\theta}_i)}{|\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i}|\bar{\theta}_i)|}.$$

Define  $\Phi = \{\phi, \phi'\}$ . All IR constraints bind because for each  $i \in I$ ,  $\eta_i$  extracts agent  $i$ 's full surplus on path, and  $c\psi_i$  has zero interim expected value under agent  $i$ 's belief. The choice of  $c$  gives any unilateral deviator a non-positive worst-case expected payoff, and thus the IC condition also holds. Hence,  $(q, \Phi)$  extracts the full surplus.

**Sufficiency of Part 2.** Pick any BB transfer rule  $\eta : \Theta \rightarrow \mathbb{R}^N$  such that

$$\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) + \eta_i(\theta_i, \theta_{-i})] p_i(\theta_{-i} | \theta_i) \geq 0$$

for all  $i \in I$  and  $\theta_i \in \Theta_i$ . For example, we can choose  $\eta_i(\theta) = \frac{1}{N} \sum_{j \in I} u_j(q(\theta), \theta) - u_i(q(\theta), \theta)$  for all  $i \in I$  and  $\theta \in \Theta$  so that all agents have equal surplus. By Lemma A.5, there exists a BB transfer rule  $\psi$  which gives all agents zero interim values on path and gives any unilateral deviator a non-zero interim expected value.

Pick any  $c$  such that  $c$  is no less than

$$\frac{\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), (\hat{\theta}_i, \theta_{-i})) + \eta_i(\hat{\theta}_i, \theta_{-i}) - u_i(q(\bar{\theta}_i, \theta_{-i}), (\bar{\theta}_i, \theta_{-i})) - \eta_i(\bar{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i)}{|\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i)|}$$

for all  $i$  and  $\bar{\theta}_i \neq \hat{\theta}_i$ , where the denominator is positive. Let  $\mathcal{M}$  be  $(q, \{\eta + c\psi, \eta - c\psi\})$ .

The IR condition follows from the choice of  $\eta$  and the fact that  $\psi$  gives agents zero interim values on path. For all  $i$  and  $\bar{\theta}_i \neq \hat{\theta}_i$ , the choice of  $c$  indicates that

$$\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), (\bar{\theta}_i, \theta_{-i})) + \eta_i(\bar{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) \geq \min\left\{ \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), (\bar{\theta}_i, \theta_{-i})) + \eta_i(\hat{\theta}_i, \theta_{-i}) \pm c\psi_i(\hat{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) \right\},$$

and thus  $\mathcal{M}$  satisfies the IC condition. The BB condition of  $\mathcal{M}$  follows from BB of  $\eta$  and  $\psi$ . Therefore,  $\mathcal{M}$  is an IR and BB mechanism with ambiguous transfers that implements  $q$ .

**Sufficiency of Part 3.** Given a profile of beliefs that can be generated by a common prior  $p$ , when the BDP property holds for all agents, the sufficiency part has been proven in Part 2. Suppose instead that there is exactly one agent,  $i$ , for whom the BDP property fails. Following a similar argument as Lemmas A.4 and A.5, one can prove that there exists a transfer rule  $\psi : \Theta \rightarrow \mathbb{R}^N$  such that

1.  $\sum_{j \in I} \psi_j(\theta) = 0$  for all  $\theta \in \Theta$ ;
2.  $\sum_{\theta_{-j} \in \Theta_{-j}} \psi_j(\theta_j, \theta_{-j}) p_j(\theta_{-j} | \theta_j) = 0$  for all  $j \in I$  and  $\theta_j \in \Theta_j$ ;
3.  $\sum_{\theta_{-j} \in \Theta_{-j}} \psi_j(\hat{\theta}_j, \theta_{-j}) p_j(\theta_{-j} | \bar{\theta}_j) \neq 0$  for all  $j \neq i$  and  $\bar{\theta}_j, \hat{\theta}_j \in \Theta_j$  satisfying  $\bar{\theta}_j \neq \hat{\theta}_j$ .



Notice that the third statement is different from the one in Lemma A.5, as the BDP property fails for agent  $i$  here.

We construct a mechanism where agent  $i$  obtains all the surplus on path. For all  $\theta \in \Theta$  and  $j \in I$  with  $j \neq i$ , let  $\eta_j(\theta) = -u_j(q(\theta), \theta_j)$ , and  $\eta_i(\theta) = -\sum_{j \neq i} \eta_j(\theta)$ .

Pick any  $c$  that is no less than

$$\max_{\substack{j \neq i, \hat{\theta}_j, \bar{\theta}_j \in \Theta_j, \\ \hat{\theta}_j \neq \bar{\theta}_j}} \frac{\sum_{\theta_{-j} \in \Theta_{-j}} [u_j(q(\hat{\theta}_j, \theta_{-j}), \bar{\theta}_j) - u_j(q(\hat{\theta}_j, \theta_{-j}), \hat{\theta}_j)] p_j(\theta_{-j} | \bar{\theta}_j)}{|\sum_{\theta_{-j} \in \Theta_{-j}} \psi_j(\hat{\theta}_j, \theta_{-j}) p_j(\theta_{-j} | \bar{\theta}_j)|}.$$

Let the set of ambiguous transfers be  $\Phi = \{\eta + c\psi, \eta - c\psi\}$ , which is IR and BB. The choice of  $\eta$ ,  $\psi$ , and  $c$  implies that agent  $j \neq i$  obtains zero MEU on path and non-positive MEU when he unilaterally misreports. Therefore,  $j$ 's IC constraints are satisfied.

For any  $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$  with  $\bar{\theta}_i \neq \hat{\theta}_i$ , the argument below verifies  $IC(\bar{\theta}_i, \hat{\theta}_i)$ :

$$\begin{aligned} & \min \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \eta_i(\bar{\theta}_i, \theta_{-i}) \pm c\psi_i(\bar{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) \right\} \\ &= \min \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\bar{\theta}_i, \theta_{-i}), \theta_j) \pm c\psi_i(\bar{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) \right\} \\ &= \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\bar{\theta}_i, \theta_{-i}), \theta_j)] p_i(\theta_{-i} | \bar{\theta}_i) \\ &\geq \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\hat{\theta}_i, \theta_{-i}), \theta_j)] p_i(\theta_{-i} | \bar{\theta}_i) \\ &\geq \min \left\{ \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\hat{\theta}_i, \theta_{-i}), \theta_j) \pm c\psi_i(\hat{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) \right\}, \end{aligned}$$

where the first equality comes from the definition of  $\eta$ , the second equality follows from the second property of  $\psi$  introduced at the beginning of this part of the proof, the first inequality comes from ex-post efficiency of  $q$  at each type profile  $(\bar{\theta}_i, \theta_{-i})$ , and the second inequality comes from the minimization operation.

Therefore, the IR and BB mechanism with ambiguous transfers implements  $q$ .  $\square$

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