Online Appendices to "Mechanism Design with Ambiguous Transfers: An Analysis in Finite Dimensional Naive Type Spaces"

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Online Appendix B presents proofs omitted from Section 4 of the paper. Online Appendix C discusses the relationship between the BDP property, Property I, and Property II.

B Omitted proofs from Section 4

Proof of Lemma 4.1. Suppose the profile of beliefs can be generated by a common prior p. Then the marginal probability $p(\theta_i) > 0$ and the conditional probability $p(\theta_{-i}|\theta_i) \equiv \frac{p(\theta)}{p(\theta_i)} = p_i(\theta_{-i}|\theta_i)$ for all $i \in I$, $\theta_i \in \Theta_i$, and $\theta_{-i} \in \Theta_{-i}$.

In this case, expression (7) in the paper can be rewritten as

$$\big(\hat{C} - \bar{C}\frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)}\big)p(\theta_{-i}|\hat{\theta}_i) = p(\theta_{-i}|\bar{\theta}_i), \forall \theta_{-i} \in \Theta_{-i}. \tag{B.1}$$

1 $\implies$ ***2***. We prove by contrapositive. Suppose the profile of beliefs does not satisfy Property I for i. Then there exist types $\bar{\theta}_i \neq \hat{\theta}_i$, a distribution $\mu \in \Delta(\Theta)$, and constants $\bar{C} > 0$, and $\hat{C} > 1$ such that expressions (6) and (7) in the paper hold. Expression (7) is equivalent to expression (B.1) under the common prior assumption. Hence, one must have $\hat{C} - \bar{C}\frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)} = 1$ and $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$. Thus, the BDP property fails for agent i.

2 $\implies$ ***3***. This step is trivial.

3 $\implies$ ***1***. We prove by contrapositive. Suppose the BDP property fails for agent i as $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$. As p is a common prior, define the prior $\mu = p$, which makes expression (6) hold. Pick any $\bar{C} \geq 1$ and define $\hat{C} = 1 + \bar{C}\frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)}$. Thus, expression (B.1) holds, which further implies that expression (7) holds. Since $\bar{C} \geq 1$, $p(\hat{\theta}_i) > 0$, and $p(\bar{\theta}_i) > 0$, we also obtain $\hat{C} > 1$. As a result, the profile of beliefs does not satisfy Property II for agent i. $\square$

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Lemma B.1: *Fix a profile of beliefs* $(p_j)_{j\in I}$*, an agent* $i \in I$*, and two types* $\bar{\theta}_i \neq \hat{\theta}_i$*. There exist coefficients* $(b_\theta)_{\theta\in\Theta}$ *and non-negative coefficients* $(a_{\theta_j})_{j\in I,\theta_j\in\Theta_j}$ *such that equation*

$$p_{\bar{\theta}_i\hat{\theta}_i} = \sum_{j\in I}\sum_{\theta_j\in\Theta_j} a_{\theta_j} p_{\theta_j\theta_j} - \sum_{\theta\in\Theta} b_\theta e_\theta \tag{B.2}$$

holds if and only if at least one of the following conditions is satisfied

1. $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$;
2. *there exists* $\mu \in \Delta(\Theta)$, $\bar{C} > 0$, *and* $\hat{C} > 1$ *such that*

$$\mu(\theta_j) > 0 \text{ and } \mu(\theta_{-j}|\theta_j) = p_j(\theta_{-j}|\theta_j) \text{ for all } (j,\theta_j) \neq (i,\hat{\theta}_i) \text{ and } \theta_{-j} \tag{B.3}$$

and

$$\hat{C} p_i(\theta_j, \cdot|\hat{\theta}_i) = p_i(\theta_j, \cdot|\bar{\theta}_i) + \bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)} p_j(\hat{\theta}_i, \cdot|\theta_j) \text{ for all } j \neq i \text{ and } \theta_j. \tag{B.4}$$

Proof. **Necessity**. Expression (B.2) implies

$$0 = a_{\theta_i} p_i(\theta_{-i}|\theta_i) - b_\theta, \forall \theta_i \neq \hat{\theta}_i, \theta_{-i}, \tag{B.5}$$

$$p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) - b_{\hat{\theta}_i,\theta_{-i}}, \forall \theta_{-i}, \tag{B.6}$$

$$0 = a_{\theta_j} p_j(\theta_{-j}|\theta_j) - b_\theta, \forall j \neq i, \theta. \tag{B.7}$$

If $N = 2$, we ignore any term θ_{-i-j} to avoid introducing additional notations.

Case 1. Suppose $a_{\tilde{\theta}_i} = 0$ for some $\tilde{\theta}_i \neq \hat{\theta}_i$. The argument below shows that $a_{\hat{\theta}_i} = 1$, $a_{\theta_j} = 0$ for all $(j,\theta_j) \neq (i,\hat{\theta}_i)$, $b_\theta = 0$ for all θ, and $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$.

Canceling $b_{\tilde{\theta}_i,\theta_{-i}}$ in (B.5) and (B.7) yields $0 = a_{\tilde{\theta}_i} p_i(\theta_{-i}|\tilde{\theta}_i) = a_{\theta_j} p_j(\tilde{\theta}_i, \theta_{-i-j}|\theta_j)$ for any agent $j \neq i$ and type $\theta_{-i} \in \Theta_{-i}$. By Assumption 2.1, it follows that $a_{\theta_j} = 0$ for any $j \neq i$ and θ_j.

By expression (B.7), the previous paragraph implies $b_\theta = 0$ for all θ. From expression (B.5), we further know $a_{\theta_i} = 0$ for all $\theta_i \neq \hat{\theta}_i$.

By canceling $b_{\hat{\theta}_i,\theta_{-i}}$ in (B.6) and (B.7), we have $a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) - p_i(\theta_{-i}|\bar{\theta}_i) = a_{\theta_j} p_j(\hat{\theta}_i, \theta_{-i-j}|\theta_j) = 0$ for all θ_{-i}. Summing the equation across all θ_{-i}, we get $a_{\hat{\theta}_i} = 1$ and thus $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$.

Case 2. Suppose $a_{\theta_i} > 0$ for all $\theta_i \neq \hat{\theta}_i$. Similar to the argument of the previous case, we know $a_{\hat{\theta}_i} > 1$ and $a_{\theta_j} > 0$ for all $(j,\theta_j) \neq (i,\hat{\theta}_i)$. Subsequently, we will establish that there exists $\mu \in \Delta(\Theta)$, $\bar{C} > 0$, and $\hat{C} > 1$ such that expressions (B.3) and (B.4) hold.

Define $\mu \in \Delta(\Theta)$ by $\mu(\theta) = \frac{b_\theta}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}}$ for all $\theta \in \Theta$. Then from expressions (B.5) and (B.7), we know $\mu(\cdot|\theta_j) = p_j(\cdot|\theta_j)$ and $\mu(\theta_j) = \frac{a_{\theta_j}}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}} > 0$ for all $(j,\theta_j) \neq (i,\hat{\theta}_i)$.

Hence, expression (B.3) holds. By canceling $b_{\hat{\theta}_i,\theta_{-i}}$ in expressions (B.6) and (B.7), we have $a_{\hat{\theta}_i} p_i(\theta_j, \cdot|\hat{\theta}_i) = p_i(\theta_j, \cdot|\bar{\theta}_i) + a_{\theta_j} p_j(\hat{\theta}_i, \cdot|\theta_j)$ for all $j \neq i$ and θ_j, where $a_{\theta_j} = \mu(\theta_j) \sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}} = \mu(\bar{\theta}_i) \frac{\mu(\theta_j|\bar{\theta}_i)}{\mu(\bar{\theta}_i|\theta_j)} \sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}} = a_{\bar{\theta}_i} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)}$. Recall $a_{\bar{\theta}_i} > 0$ and $a_{\hat{\theta}_i} > 1$. Thus, by defining $\bar{C} = a_{\bar{\theta}_i}$ and $\hat{C} = a_{\hat{\theta}_i}$, expression (B.4) holds.

Sufficiency. When $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, define $a_{\hat{\theta}_i} = 1$, $a_{\theta_j} = 0$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$, and $b_\theta = 0$ for all $\theta \in \Theta$. When expressions (B.3) and (B.4) hold for some μ, $\bar{C} > 0$, and $\hat{C} > 1$, define $a_{\bar{\theta}_i} = \bar{C}$, $a_{\hat{\theta}_i} = \hat{C}$, $b_\theta = \bar{C}\frac{\mu(\theta)}{\mu(\bar{\theta}_i)}$ for all $\theta \in \Theta$, and $a_{\theta_k} = \bar{C}\frac{\mu(\theta_k)}{\mu(\bar{\theta}_i)}$ for all $(k, \theta_k) \neq (i, \bar{\theta}_i), (i, \hat{\theta}_i)$. For both cases, it is easy to verify expression (B.2). □

Lemma B.2: *Given a profile of beliefs $(p_j)_{j\in I}$ and an agent $i \in I$, if both the BDP and Property I are satisfied for agent i, then for each pair of $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ with $\bar{\theta}_i \neq \hat{\theta}_i$, there exists a transfer rule $\psi^{\bar{\theta}_i\hat{\theta}_i} : \Theta \to \mathbb{R}^N$ such that*

1. $\sum_{j\in I} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta) = 0$ *for all* $\theta \in \Theta$;
2. $\sum_{\theta_{-j}\in\Theta_{-j}} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j}|\theta_j) \geq 0$ *for all* $j \in I$, $\theta_j \in \Theta_j$;
3. $\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i}|\bar{\theta}_i) < 0.$

Proof. We prove by contraposition. Suppose there exists a pair of $\bar{\theta}_i \neq \hat{\theta}_i$ such that no $\psi^{\bar{\theta}_i\hat{\theta}_i}$ satisfies these conditions. By Motzkin's theorem of the alternative, there exist coefficients $(b_\theta)_{\theta\in\Theta}$ and non-negative coefficients $(a_{\theta_j})_{j\in I,\theta_j\in\Theta_j}$ such that expression (B.2) holds. By Lemma B.1 and the definition of Property I, the BDP property or Property I should fail for agent i. □

Lemma B.3: *Given a profile of beliefs $(p_i)_{i\in I}$, if the BDP property holds for all agents, then the profile of beliefs satisfies Property I for at least $N - 1$ agents.*

Proof. Let all agents satisfy the BDP property. Suppose by way of contradiction that there are agents $i \neq j$ such that profile of beliefs does not satisfy Property I for i and j. In view of Definition 4.1 and the two-case argument in the proof of Lemma B.1, there exist types $\bar{\theta}_i \neq \hat{\theta}_i$, $\bar{\theta}_j \neq \hat{\theta}_j$, coefficients $(a_{\theta_k} > 0)_{k\in I,\theta_k\in\Theta_k}$ where $a_{\hat{\theta}_i} > 1$, $(b_\theta)_{\theta\in\Theta}$, $(c_{\theta_k} > 0)_{k\in I,\theta_k\in\Theta_k}$ where $c_{\hat{\theta}_j} > 1$, and $(d_\theta)_{\theta\in\Theta}$, such that $p_{\bar{\theta}_i\hat{\theta}_i} = \sum_{k\in I}\sum_{\theta_k\in\Theta_k} a_{\theta_k} p_{\theta_k\theta_k} - \sum_{\theta\in\Theta} b_\theta e_\theta$ and $p_{\bar{\theta}_j\hat{\theta}_j} = \sum_{k\in I}\sum_{\theta_k\in\Theta_k} c_{\theta_k} p_{\theta_k\theta_k} - \sum_{\theta\in\Theta} d_\theta e_\theta$. Thus, the following equations hold.

$$
\begin{aligned}
&0 = a_{\theta_i} p_i(\theta_{-i}|\theta_i) - b_\theta, \forall\theta_i \neq \hat{\theta}_i \text{ and } \theta_{-i}, && 0 = c_{\theta_j} p_j(\theta_{-j}|\theta_j) - d_\theta, \forall\theta_j \neq \hat{\theta}_j \text{ and } \theta_{-j},\\
&p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) - b_{\hat{\theta}_i,\theta_{-i}}, \forall\theta_{-i}, && p_j(\theta_{-j}|\bar{\theta}_j) = c_{\hat{\theta}_j} p_j(\theta_{-j}|\hat{\theta}_j) - d_{\hat{\theta}_j,\theta_{-j}}, \forall\theta_{-j},\\
&0 = a_{\theta_j} p_j(\theta_{-j}|\theta_j) - b_\theta, \forall\theta, && 0 = c_{\theta_i} p_i(\theta_{-i}|\theta_i) - d_\theta, \forall\theta.
\end{aligned}
$$

In the argument below, we ignore θ_{-i-j} if $N = 2$ to avoid introducing additional notations. Canceling all b_θ, d_θ, and $p_j(\theta_{-j}|\theta_j)$ in the above equations yields:

$$[\frac{a_{\theta_i}}{a_{\theta_j}} - \frac{c_{\theta_i}}{c_{\theta_j}}]p_i(\theta_{-i}|\theta_i) = 0, \forall \theta_i \neq \hat{\theta}_i, \theta_j \neq \hat{\theta}_j, \text{ and } \theta_{-i-j}, \tag{B.8}$$

$$[\frac{a_{\hat{\theta}_i}}{a_{\theta_j}} - \frac{c_{\hat{\theta}_i}}{c_{\theta_j}}]p_i(\theta_{-i}|\hat{\theta}_i) = \frac{p_i(\theta_{-i}|\bar{\theta}_i)}{a_{\theta_j}}, \forall \theta_j \neq \hat{\theta}_j \text{ and } \theta_{-i-j}, \tag{B.9}$$

$$[\frac{a_{\theta_i}}{a_{\hat{\theta}_j}} - \frac{c_{\theta_i}}{c_{\hat{\theta}_j}}]p_i(\hat{\theta}_j, \theta_{-i-j}|\theta_i) = \frac{c_{\theta_i} p_i(\bar{\theta}_j, \theta_{-i-j}|\theta_i)}{c_{\hat{\theta}_j} c_{\bar{\theta}_j}}, \forall \theta_i \neq \hat{\theta}_i \text{ and } \theta_{-i-j}, \tag{B.10}$$

$$[\frac{a_{\hat{\theta}_i}}{a_{\hat{\theta}_j}} - \frac{c_{\hat{\theta}_i}}{c_{\hat{\theta}_j}}]p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{a_{\hat{\theta}_j}} + \frac{c_{\hat{\theta}_i} p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}{c_{\hat{\theta}_j} c_{\bar{\theta}_j}}, \forall \theta_{-i-j}. \tag{B.11}$$

Step 1. We want to prove that for each $\theta_{-i-j} \in \Theta_{-i-j}$, either all the four numbers $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, $p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$, $p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, and $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$ are positive, or they are all equal to zero.

By Assumption 2.1, there exists $\tilde{\theta}_{-i-j}$ such that $p_i(\bar{\theta}_j, \tilde{\theta}_{-i-j}|\bar{\theta}_i) > 0$. Hence, expressions (B.9) and (B.10) imply $\frac{a_{\hat{\theta}_i}}{a_{\bar{\theta}_j}} - \frac{c_{\hat{\theta}_i}}{c_{\bar{\theta}_j}}, \frac{a_{\bar{\theta}_i}}{a_{\hat{\theta}_j}} - \frac{c_{\bar{\theta}_i}}{c_{\hat{\theta}_j}} > 0$. Thus, for each θ_{-i-j}, either $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, $p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$, $p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$, or $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) = p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) = 0$.

In the previous case, expression (B.11) implies that we also have $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) > 0$.

In the latter case, we must have $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = 0$, because otherwise expression (B.11) would imply $\frac{a_{\hat{\theta}_i}}{a_{\hat{\theta}_j}} = \frac{c_{\hat{\theta}_i}}{c_{\hat{\theta}_j}}$, which would further imply that $p_i(\hat{\theta}_j, \cdot|\bar{\theta}_i) = p_i(\bar{\theta}_j, \cdot|\hat{\theta}_i) = \mathbf{0}$, a contradiction to Assumption 2.1.

Step 2. We want to prove that for each $\theta_{-i-j} \in \Theta_{-i-j}$ such that $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$,

$$\frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)} = \frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}.$$

When $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$, canceling $a_{\hat{\theta}_j}$, $c_{\hat{\theta}_j}$, and $c_{\hat{\theta}_i}$ in expressions (B.8) through (B.11) yields

$$\frac{c_{\bar{\theta}_j} p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) + p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{c_{\bar{\theta}_j} p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) + p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} = \frac{a_{\hat{\theta}_i} - \frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}}{a_{\hat{\theta}_i} - \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}} \times \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}.$$

From the two-case argument in Lemma B.1 and the definition of Property I, we know that $a_{\hat{\theta}_i} - \frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} = a_{\bar{\theta}_i} \frac{p_i(\bar{\theta}_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\bar{\theta}_j)} \frac{p_j(\hat{\theta}_i, \theta_{-i-j}|\bar{\theta}_j)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} \geq 0$. Similarly, $a_{\hat{\theta}_i} - \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} \geq 0$. Suppose $\frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} > (<) \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}$. The left-hand side of the above equation is greater (less) than $\frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}$ and the right-hand side is less (greater) than $\frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}$, a contradiction. Hence, $\frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} = \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}$. Rearranging terms yields the desired result.

Step 3. We want to prove that $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, which would contradict the BDP property of agent i and complete the proof.

Expression (B.8) implies that $\frac{a_{\bar{\theta}_i}}{a_{\theta_j}} = \frac{c_{\bar{\theta}_i}}{c_{\theta_j}}$ for all $\theta_j \neq \hat{\theta}_j$. Plugging it into expression (B.9) yields $(\frac{a_{\hat{\theta}_i}}{a_{\bar{\theta}_i}} - \frac{c_{\hat{\theta}_i}}{c_{\bar{\theta}_i}})p_i(\theta_{-i}|\hat{\theta}_i) = \frac{1}{a_{\bar{\theta}_i}}p_i(\theta_{-i}|\bar{\theta}_i)$ for all $\theta_j \neq \hat{\theta}_j$ and θ_{-i-j}. Hence,

$$\frac{p_i(\theta_j, \tilde{\theta}_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)} = \frac{p_i(\theta_j, \tilde{\theta}_{-i-j}|\hat{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}, \forall \theta_j \neq \hat{\theta}_j, \theta_{-i-j} \text{ s.t. } p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0, \text{ and } \tilde{\theta}_{-i-j}.$$

Combining this expression with Step 1 and Step 2, we have established the desired result. □

Lemma B.4: *Let q be an efficient allocation rule under a private value environment. For any $i \in I$, $\tilde{\Theta}_i \subseteq \Theta_i$ with $|\tilde{\Theta}_i| \geq 2$, and distribution $\pi_i \in \Delta(\Theta_{-i})$, there exist values $(U_{\theta_i} \geq 0)_{\theta_i \in \tilde{\Theta}_i}$ such that $U_{\theta_i} - U_{\theta_i'} \geq \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\theta_i', \theta_{-i}), \theta_i) - u_i(q(\theta_i', \theta_{-i}), \theta_i')]\pi_i(\theta_{-i})$ for all $\theta_i, \theta_i' \in \tilde{\Theta}_i$.*

Proof. Let a loop be a finite sequence $(\theta_i^1, \theta_i^2, ..., \theta_i^K)$ in $\tilde{\Theta}_i$ with length $K \geq 2$ and $\theta_i^1 = \theta_i^K$. As q is ex-post efficient, $u_i(q(\theta_i^{k+1}, \theta_{-i}), \theta_i^{k+1}) + \sum_{j \neq i} u_j(q(\theta_i^{k+1}, \theta_{-i}), \theta_j) \geq u_i(q(\theta_i^k, \theta_{-i}), \theta_i^{k+1}) + \sum_{j \neq i} u_j(q(\theta_i^k, \theta_{-i}), \theta_j)$ for all $k = 1, ..., K-1$ and $\theta_{-j} \in \Theta_{-j}$. Summing the inequalities across $k = 1, ..., K-1$ and taking into account $\theta_i^1 = \theta_i^K$, we obtain that $\sum_{k=1}^{K-1} [u_i(q(\theta_i^k, \theta_{-i}), \theta_i^{k+1}) - u_i(q(\theta_i^k, \theta_{-i}), \theta_i^k)] \leq 0$. This is the "cyclical monotonicity" condition is the literature.

Fix an arbitrary $\tilde{\theta}_i \in \tilde{\Theta}_i$. For each $(\theta_i, \theta_{-i}) \in \tilde{\Theta}_i \times \Theta_{-i}$, define the function $V_i(\cdot) : \tilde{\Theta}_i \times \Theta_{-i} \to \mathbb{R}$ by:

$$V_i(\theta_i, \theta_{-i}) \equiv \sup_{\substack{(\theta_i^1, ..., \theta_i^k) \text{ is any finite sequence} \\ \text{starting with } \tilde{\theta}_i \text{ and ending with } \theta_i}} \sum_{k=1}^{K-1} [u_i(q(\theta_i^k, \theta_{-i}), \theta_i^{k+1}) - u_i(q(\theta_i^k, \theta_{-i}), \theta_i^k)].$$

Then by Theorem 1 of Rochet (1987) or Proposition 5.2 of Börgers et al. (2015), $V_i(\cdot)$ is a well-defined function satisfying

$$V_i(\theta_i, \theta_{-i}) - V_i(\theta_i', \theta_{-i}) \geq \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\theta_i', \theta_{-i}), \theta_i) - u_i(q(\theta_i', \theta_{-i}), \theta_i'), \forall \theta_i, \theta_i' \in \tilde{\Theta}_i.$$

Choose a sufficiently large $C > 0$. Then, $U_{\theta_i} \equiv \sum_{\theta_{-i} \in \Theta_{-i}} V_i(\theta_i, \theta_{-i})\pi_i(\theta_{-i}) + C$ is non-negative for all $\theta_i \in \tilde{\Theta}_i$. The values $(U_{\theta_i} \geq 0)_{\theta_i \in \tilde{\Theta}_i}$ satisfy the desired condition. □

***Proof of Proposition 4.1*.** **Necessity of Part 1**. Suppose there exists an agent $i \in I$ for whom the BDP property fails. The same construction as in the necessity proof of Parts 1 and 2 of Theorem 3.1 can establish the necessity of the BDP property.

To prove the necessity of Property II, suppose there exists $i \in I$, $\bar{\theta}_i \neq \hat{\theta}_i$, $\mu \in \Delta(\Theta)$, $\bar{C} \geq 1$, and $\hat{C} > 1$ such that expressions (8) and (9) in Definition 4.1 hold so that the profile of beliefs does not satisfy Property II for agent i. Fix any agent $j \neq i$. Consider the same profile of utility functions as in the necessity proof of Part 2 of Theorem 3.1 except that a and B satisfy $0 < [\hat{C} + \bar{C}\sum_{(k,\theta_k)\neq(i,\hat{\theta}_i)} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)} \frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)} - 1]a < B$. Suppose by way of contradiction that an IR and BB mechanism with ambiguous transfers (q, Φ) implements q. Following the same argument when deriving expressions (1) and (2) of Example 4.1, we know for all $\epsilon > 0$ there exists $\phi \in \Phi$ such that:

$$IR(\theta_k) \qquad \sum_{\theta_{-k\in\Theta_{-k}}} \phi_k(\theta_k, \theta_{-k})p_k(\theta_{-k}|\theta_k) \geq -a, \forall k \in I \text{ and } \theta_k \in \Theta_k$$

$$BB(\theta) \qquad -\sum_{k\in I} \phi_k(\theta) = 0, \forall \theta \in \Theta$$

$$IC(\bar{\theta}_i\hat{\theta}_i) \qquad \sum_{\theta_{-i\in\Theta_{-i}}} \phi_i(\bar{\theta}_i, \theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) - \sum_{\theta_{-i\in\Theta_{-i}}} \phi_i(\hat{\theta}_i, \theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) + \epsilon \geq B.$$

Multiply $IR(\bar{\theta}_i)$ by $\bar{C} - 1$, $IR(\hat{\theta}_i)$ by $\hat{C}$, each $IR(\theta_k)$ where $(k, \theta_k) \neq (i, \bar{\theta}_i), (i, \hat{\theta}_i)$ by $\bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)} \frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)}$, each $BB(\theta)$ by $\bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)p_j(\theta_{-j}|\theta_j)}{p_j(\bar{\theta}_i|\theta_j)}$, and $IC(\bar{\theta}_i\hat{\theta}_i)$ by 1. Add up and let ϵ go to zero. We have $0 \geq B - [\hat{C} + \bar{C}\sum_{(k,\theta_k)\neq(i,\hat{\theta}_i)} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)} \frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)} - 1]a > 0$, a contradiction.

Necessity of Part 2. By relabeling the indices, assume without loss of generality that the BDP property fails for agent 2 and that the profile of beliefs does not satisfy Property II for agent 1. In particular, assume $p_2(\cdot|\theta_2^1) = p_2(\cdot|\theta_2^2)$ and that for $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^2, \theta_1^1)$, there exists $\mu \in \Delta(\Theta)$, $\bar{C} \geq 1$, and $\hat{C} > 1$ such that expressions (8) and (9) hold.

We now construct a profile of private value utility functions $(u_i)_{i\in I}$ and an efficient allocation rule q that is not implementable via a mechanism with ambiguous transfers. In this way, we can prove the necessity statement in Part 2 of Proposition 4.1.

Following the necessity proof of Part 3 of Theorem 3.1, we define $(v_i(\theta_i))_{i\in I,\theta_i\in\Theta_i}$, $(u_i)_{i\in I}$, and q in the same way, except for a minor difference in ranking within $(v_i(\theta_i))_{i\in I,\theta_i\in\Theta_i}$. Here, we let the parameters satisfy $v_2(\theta_2^1) > -v_1(\theta_1^1) > v_2(\theta_2^2) > -v_1(\theta_1^2) > ... > v_2(\theta_2^{\min\{|\Theta_1|,|\Theta_2|\}}) > -v_1(\theta_1^{\min\{|\Theta_1|,|\Theta_2|\}})$. When $|\Theta_2| \geq |\Theta_1|$, we further let $-v_1(\theta_1^{|\Theta_1|\}}) > v_i(\theta_i) > 0$ for any pair of $i \neq 1$ and θ_i not already in the ranking. When $|\Theta_2| < |\Theta_1|$, we let $-v_1(\theta_1^{|\Theta_2|}) > -v_1(\theta_1) > v_i(\theta_i)$ for any θ_1 not in the ranking and any pair of $i \neq 2$ and θ_i not in the ranking.

Suppose by contradiction that q is implementable by an IR and BB mechanism with ambiguous transfers (q, Φ). Hence, for each $i \in I$ and $\theta_i \in \Theta_i$, the MEU of participation is $U_{\theta_i} \geq 0$. Following the same argument when deriving expressions (1) and (2) of Example 4.1, we know that for all $\epsilon > 0$, there exists a transfer rule $\phi \in \Phi$ such that

$$IR(\theta_i) \qquad \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_i, \theta_{-i})p_i(\theta_{-i}|\theta_i) \geq U_{\theta_i} - \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i, \theta_{-i}), \theta_i)p_i(\theta_{-i}|\theta_i), \forall i \in I, \theta_i \in \Theta_i,$$

$$BB(\theta) \qquad -\sum_{i\in I} \phi_i(\theta) = 0, \forall \theta \in \Theta,$$

$$IC(\theta_1^2\theta_1^1) \quad U_{\theta_1^2}+\epsilon \geq \sum_{\theta_{-1}\in\Theta_{-1}} \phi_1(\theta_1^1,\theta_{-1})p_1(\theta_{-1}|\theta_1^2) + \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2).$$

Multiply $IR(\theta_1^1)$ by $\hat{C}$, each $IR(\theta_i)$ where $(i,\theta_i)\neq(1,\theta_1^1)$ by $\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)}$, each $BB(\theta)$ by $\bar{C}\frac{p_1(\theta_2|\theta_1^2)p_2(\theta_{-2}|\theta_2)}{p_2(\theta_1^2|\theta_2)}$, and $IC(\theta_1^2\theta_1^1)$ by 1. Add them up and let ϵ go to zero. Since the profile of beliefs does not satisfy Property II for agent 1, we have

$$\begin{aligned} U_{\theta_1^2} \geq & \sum_{(i,\theta_i)\neq(1,\theta_1^1)} [U_{\theta_i} - \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\theta_i)]\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)} \\ & + [U_{\theta_1^1} - \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^1)p_1(\theta_{-1}|\theta_1^1)]\hat{C} + \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2). \quad \text{(B.12)} \end{aligned}$$

Following the method to derive expressions (18) and (19), we know from $IC(\theta_2^1\theta_2^2)$ and $p_2(\cdot|\theta_2^1)=p_2(\cdot|\theta_2^2)$ that

$$U_{\theta_2^1} \geq U_{\theta_2^2} + \sum_{\theta_{-2}\in\Theta_{-2}} [u_2(q(\theta_2^2,\theta_{-2}),\theta_2^1)) - u_2(q(\theta_2^2,\theta_{-2}),\theta_2^2)]p_2(\theta_{-2}|\theta_2^1).$$

Plug the above expression into expression (B.12). Since the coefficient of $IR(\theta_1^2)$ is $\bar{C}\geq 1$ and each $U_{\theta_i}\geq 0$, we have

$$\begin{aligned} 0 \geq & -\sum_{(i,\theta_i)\neq(1,\theta_1^1),(2,\theta_2^1)} [\sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\theta_i)]\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)} \\ & + \sum_{\theta_{-2}\in\Theta_{-2}} [u_2(q(\theta_2^2,\theta_{-2}),\theta_2^1) - u_2(q(\theta_2^2,\theta_{-2}),\theta_2^2) - u_2(q(\theta_2^1,\theta_{-2}),\theta_2^1)]p_2(\theta_{-2}|\theta_2^1)\bar{C}\frac{p_1(\theta_2^1|\theta_1^2)}{p_2(\theta_1^2|\theta_2^1)} \\ & - \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^1)p_1(\theta_{-1}|\theta_1^1)\hat{C} + \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2). \end{aligned}$$

We can plug the explicit forms of q and $(u_i)_{i\in I}$ as well as expression (9) into the above expression so that the new inequality is written in terms of $(v_i(\theta_i))_{i\in I,\theta_i\in\Theta_i}$. On the right-hand side, the coefficient of $v_2(\theta_2^1)$ is $-\bar{C}p_2(\theta_1^1|\theta_2^1)\frac{p_1(\theta_2^1|\theta_1^2)}{p_2(\theta_1^2|\theta_2^1)}$, and the coefficient of $v_1(\theta_1^1)$ can be expressed into $-p_1(\theta_2^1|\theta_1^2) - \bar{C}p_2(\theta_1^1|\theta_2^1)\frac{p_1(\theta_2^1|\theta_1^2)}{p_2(\theta_1^2|\theta_2^1)}$. By letting $-v_1(\theta_1^1)$ and $v_2(\theta_2^1)$ approach each other and letting all other $v_i(\theta_i)$ approach zero, we know that the right-hand side approaches $-v_1(\theta_1^1)p_1(\theta_2^1|\theta_1^2)>0$, a contradiction.

Therefore, q is not implementable via an IR and BB mechanism with ambiguous transfers.

Sufficiency of Part 1. Suppose the profile of beliefs satisfies the BDP property and Property I for all agents. For each i and $\bar{\theta}_i\neq\hat{\theta}_i$, there exists $\psi^{\bar{\theta}_i\hat{\theta}_i}:\Theta\to\mathbb{R}^N$ satisfying the conditions stated in Lemma B.2.

Let η be any IR and BB transfer rule. Define $\Phi=\{\eta,\eta+c\psi^{\bar{\theta}_j\hat{\theta}_j}: j\in I,\bar{\theta}_j,\hat{\theta}_j\in\Theta_j,\bar{\theta}_j\neq\hat{\theta}_j\}$, where c is sufficiently large such that for all $j\in I$ and $\bar{\theta}_j\neq\hat{\theta}_j$, the expression $\sum_{\theta_{-j}\in\Theta_{-j}}[u_j(q(\hat{\theta}_j,\theta_{-j}),(\bar{\theta}_j,\theta_{-j})) - u_j(q(\bar{\theta}_j,\theta_{-j}),(\bar{\theta}_j,\theta_{-j})) + \eta_j(\hat{\theta}_j,\theta_{-j}) - \eta_j(\bar{\theta}_j,\theta_{-j}) + c\psi_j^{\bar{\theta}_j\hat{\theta}_j}(\hat{\theta}_j,\theta_{-j})]p_j(\theta_{-j}|\bar{\theta}_j)$ is non-positive.

For any type-$\bar{\theta}_i$ agent i, the inequality below shows that misreporting $\hat{\theta}_i$ is not profitable:

$$
\begin{aligned}
&\min_{\phi\in\Phi} \sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})) + \phi_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
=&\sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})) + \eta(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
\geq&\sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})) + \eta(\hat{\theta}_i,\theta_{-i}) + c\psi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
\geq&\min_{\phi\in\Phi} \sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})) + \phi_i(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i),
\end{aligned}
$$

where the equality follows from the second condition in the statement of Lemma B.2 and the composition of ambiguous transfers, the first inequality comes from the choice of c, and the second inequality comes from the composition of ambiguous transfers again. The conditions of IR and BB are easy to check.

Sufficiency of Part 2. When beliefs can be generated by a common prior, in view of Lemma 4.1, we have established the desired result. Hence, we focus on the case when beliefs cannot be generated by a common prior. Suppose there do not exist agents $i \neq j$ such that the profile of beliefs fails to satisfy Property I for i and the BDP property fails to hold for j. Then one of the following two cases is true.

Case 1: there are at least $N-1$ agents for whom the BDP property and Property I are both satisfied. By Lemma B.3, a special situation is that the BDP property holds for all agents.

By Lemma B.2, there exists $I' \subseteq I$ with $|I'| \geq N-1$ such that for each $i \in I'$ and $\bar{\theta}_i \neq \hat{\theta}_i$, there exists $\psi^{\bar{\theta}_i\hat{\theta}_i} : \Theta \to \mathbb{R}^N$ satisfying the three conditions in the lemma.

Pick an agent $i \in I$, where $\{i\} = I\backslash I'$ if $I\backslash I'$ is a singleton and $i \in I$ is arbitrary if $I\backslash I' = \emptyset$. As in the proof of Part 3 of Theorem 3.1, let η be an IR and BB transfer rule such that agent i obtains all the surplus. Define $\Phi = \{\eta\} \cup \{\eta + c\psi^{\bar{\theta}_j\hat{\theta}_j} : j \in I, j \neq i, \bar{\theta}_j, \hat{\theta}_j \in \Theta_j, \bar{\theta}_j \neq \hat{\theta}_j\}$, where c is sufficiently large such that for all $j \neq i$ and $\bar{\theta}_j \neq \hat{\theta}_j$,

$$0 \geq \sum_{\theta_{-j}\in\Theta_{-j}} [u_j(q(\hat{\theta}_j,\theta_{-j}),\bar{\theta}_j) - u_j(q(\hat{\theta}_j,\theta_{-j}),\hat{\theta}_j) + c\psi_j^{\bar{\theta}_j\hat{\theta}_j}(\hat{\theta}_j,\theta_{-j})]p_j(\theta_{-j}|\bar{\theta}_j).$$

For agent $j \neq i$ with type θ_j, truthfully reporting gives him the MEU level of zero because his worst transfer rule, η, extracts all his surplus. Thus, agent j's IR constraints bind. The choice of c makes misreporting unprofitable, which establishes his IC constraints.

When all agents truthfully report, the MEU of a type-$\bar{\theta}_i$ agent i is

$$\min_{\phi\in\Phi} \sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\bar{\theta}_i,\theta_{-i}),\bar{\theta}_i) + \phi_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)$$

$$= \sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\bar{\theta}_i,\theta_{-i}),\bar{\theta}_i) + \sum_{j\neq i} u_j(q(\bar{\theta}_i,\theta_{-i}),\theta_j)]p_i(\theta_{-i}|\bar{\theta}_i).$$

By the ex-post efficiency of the allocation rule q, the value of the above expression is non-negative, and thus agent i's IR constraints hold. Also by efficiency of q, the value of the above expression is weakly higher than $\sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i) + \sum_{j\neq i} u_j(q(\hat{\theta}_i,\theta_{-i}),\theta_j)]p_i(\theta_{-i}|\bar{\theta}_i)$ for all $\hat{\theta}_i \neq \bar{\theta}_i$. Note the latter expression is weakly larger than his MEU of misreporting $\hat{\theta}_i$,

$$\min_{\phi\in\Phi} \sum_{\theta_{-i}\in\Theta_{-i}} [u_i(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i) + \phi_i(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i).$$

Hence, we have also verified agent i's IC constraints.

The BB condition is easy to see. Therefore, the IR and BB mechanism with ambiguous transfers implements q.

Case 2: the profile of beliefs satisfies Property I for all agents.

For any $j \in I$, let $\mathcal{P}_j$ be the partition of Θ_j such that $p_j(\cdot|\theta_j) = p_j(\cdot|\theta_j')$ if and only if θ_j and θ_j' are in the same $\tilde{\Theta}_j \in \mathcal{P}_j$. For each $\tilde{\Theta}_j$ with $|\tilde{\Theta}_j| \geq 2$ and $\theta_j \in \tilde{\Theta}_j$, define U_{θ_j} according to Lemma B.4. For a singleton $\tilde{\Theta}_j \in \mathcal{P}_j$ and $\{\theta_j\} = \tilde{\Theta}_j$, define $U_{\theta_j} = 0$.

We will demonstrate that for each agent i and each pair of $\bar{\theta}_i \neq \hat{\theta}_i$, the following system has a solution $\phi^{\bar{\theta}_i\hat{\theta}_i}$.

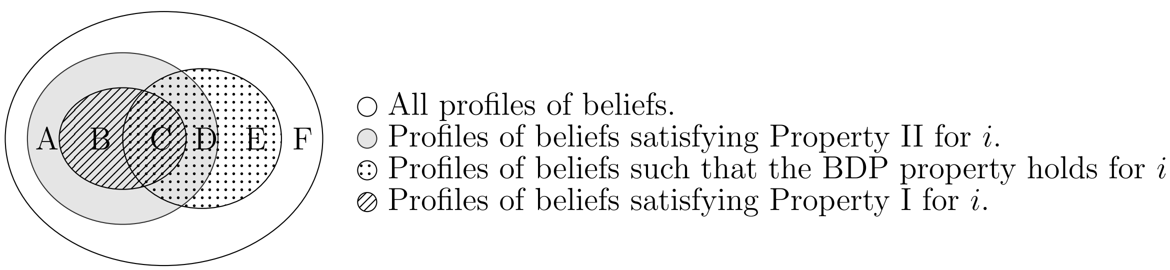
$$\begin{aligned}
&IR(\bar{\theta}_i) \quad \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^{\bar{\theta}_i\hat{\theta}_i}(\bar{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) = U_{\bar{\theta}_i} - \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\bar{\theta}_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\bar{\theta}_i),\\
&IR(\theta_j) \quad \sum_{\theta_{-j}\in\Theta_{-j}} \phi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta_j,\theta_{-j})p_j(\theta_{-j}|\theta_j) \geq U_{\theta_j} - \sum_{\theta_{-j}\in\Theta_{-j}} u_j(q(\theta_j,\theta_{-j}),\theta_j)p_j(\theta_{-j}|\theta_j), \forall (j,\theta_j) \neq (i,\bar{\theta}_i),\\
&BB(\theta) \quad\quad -\sum_{j\in I} \phi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta) = 0, \forall\theta\in\Theta,\\
&IC(\bar{\theta}_i\hat{\theta}_i) \quad -\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) \geq -U_{\bar{\theta}_i} + \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i)p_i(\theta_{-i}|\bar{\theta}_i).
\end{aligned}$$

Consider any group of coefficients $a_{\bar{\theta}_i}$ of $IR(\bar{\theta}_i)$, $a_{\theta_j} \geq 0$ of $IR(\theta_j)$ for each $(j,\theta_j) \neq (i,\bar{\theta}_i)$, b_θ of $BB(\theta)$ for each $\theta\in\Theta$, and $\gamma_{\bar{\theta}_i\hat{\theta}_i} \geq 0$ of $IC(\bar{\theta}_i\hat{\theta}_i)$ satisfying $\left((a_{\theta_j})_{j\in I,\theta_j\in\Theta_j}, (b_\theta)_{\theta\in\Theta}, \gamma_{\bar{\theta}_i\hat{\theta}_i}\right) \neq \mathbf{0}$, such that the weighted sum of the left-hand sides of the expressions is zero. We now prove that the weighted sum of the right-hand sides of the expressions is always non-positive. First notice that $\gamma_{\bar{\theta}_i\hat{\theta}_i} \neq 0$. Otherwise, since the vector of coefficients is non-zero, $a_{\theta_j} \geq 0$ for each $(j,\theta_j) \neq (i,\bar{\theta}_i)$, and Assumption B.2 holds, it is easy to prove that $a_{\theta_j} > 0$ and $\sum_{\theta_{-j}} b_{\theta_j,\theta_{-j}} > 0$ for all $j \in I$ and $\theta_j \in \Theta_j$, and that $b_\theta \geq 0$ for all $\theta\in\Theta$. Then, μ defined by $\mu(\theta) = \frac{b_\theta}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}}$ for all θ is a common prior, contradicting the assumption that beliefs cannot be generated by a common prior. Hence, it must be the case that $\gamma_{\bar{\theta}_i\hat{\theta}_i} > 0$. From Lemma B.2 and the fact that Property I is satisfied for all agents, we know: (1) $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, and (2) among all the coefficients, $a_{\hat{\theta}_i} = \gamma_{\bar{\theta}_i\hat{\theta}_i} > 0$ and everything else is zero. According to Lemma B.4, the choice of $U_{\bar{\theta}_i}$ and $U_{\hat{\theta}_i}$ satisfies $U_{\hat{\theta}_i} - U_{\bar{\theta}_i} + \sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) - u_i(q(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)]p_i(\theta_{-i}|\bar{\theta}_i) \leq 0$. Hence, the weighted sum of the right-hand sides is non-positive. Then, Gale's theorem of the alternative can be used to guarantee the existence of a solution $\phi^{\bar{\theta}_i\hat{\theta}_i}$.

Therefore, for each agent i and each pair of $\bar{\theta}_i \neq \hat{\theta}_i$, the system has a solution $\phi^{\bar{\theta}_i\hat{\theta}_i}$. Let the set of ambiguous transfers be $\Phi = \{\phi^{\bar{\theta}_i\hat{\theta}_i}, \forall i, \bar{\theta}_i, \hat{\theta}_i \in \Theta_i, \bar{\theta}_i \neq \hat{\theta}_i\}$. It is easy to see that (q, Φ) is an IR and BB mechanism with ambiguous transfers that implements q. □

C Relationship between the BDP property, Property I, and Property II

In this section, we explore the relationship between the BDP property, Property I, and Property II when beliefs may not necessarily be generated from a common prior. Fix an agent i. The diagram below summarizes the relationship between the three properties.



All profiles of beliefs $(p_j)_{j\in I}$ are in the largest ellipse. All profiles of beliefs satisfying Property I for agent i are in the line-patterned zone. Those satisfying Property II for agent i are in the shaded region. All profiles of beliefs such that the BDP property holds for agent i are in the dotted ellipse.[1] The diagram partitions the set of all profiles of beliefs into six sets: A, B, C, D, E, and F. For example, a profile of beliefs in A satisfies Property II for agent i, but not Property I or the BDP property for agent i.

We have shown in Lemma 4.1 that when a profile of beliefs can be generated by a common prior, it satisfies Property I for agent i, if and only if it satisfies Property II for agent i, if and only if the BDP property holds for agent i. Thus, a profile of beliefs that can be generated by a common prior either falls in Set C (where all the three properties are satisfied for agent i) or Set F (where all the three properties fail for agent i).

However, when a profile of beliefs cannot be generated by a common prior, it may fall in the other sets. For example, we can consider a two-agent environment where agent 1 has

[1] This only requires p_i to satisfy the BDP property. Other agents' beliefs can be viewed as free variables.

three types and 2 has two types. Let $i = 1$. We will construct profiles of beliefs that fall in each of the six sets.

Set A: The beliefs are given below. Apparently, the BDP property fails for agent 1.

$p_1(\theta_2 \mid \theta_1)$	θ_2^1	θ_2^2
θ_1^1	$\frac{3}{7}$	$\frac{4}{7}$
θ_1^2	$\frac{4}{7}$	$\frac{3}{7}$
θ_1^3	$\frac{4}{7}$	$\frac{3}{7}$

$p_2(\theta_1 \mid \theta_2)$	θ_2^1	θ_2^2
θ_1^1	$\frac{1}{9}$	$\frac{1}{2}$
θ_1^2	$\frac{4}{9}$	$\frac{1}{4}$
θ_1^3	$\frac{4}{9}$	$\frac{1}{4}$

$\mu(\theta)$	θ_2^1	θ_2^2
θ_1^1	$\frac{1}{21}$	$\frac{6}{21}$
θ_1^2	$\frac{4}{21}$	$\frac{3}{21}$
θ_1^3	$\frac{4}{21}$	$\frac{3}{21}$

The profile of beliefs does not satisfy Property I for agent 1, because with types $(\bar{\theta}_i, \hat{\theta}_i) = (\theta_1^2, \theta_1^1)$, the distribution μ defined above, and constants $\bar{C} = 0.5$ and $\hat{C} = 1.5$, expressions (6) and (7) hold.

The profile of beliefs satisfies Property II for agent 1. One may conjecture that $(\bar{\theta}_i, \hat{\theta}_i) = (\theta_1^2, \theta_1^1)$ or (θ_1^3, θ_1^1) in Definition 4.1. However, to satisfy expression (9), it must follow that $\bar{C} < 1$ in both cases. One may also conjecture that $\hat{\theta}_i = \theta_1^2$ or θ_1^3. However, in either case, there does not exist a prior μ such that expression (8) holds. Hence, there do not exist types $\bar{\theta}_i, \hat{\theta}_i$, a prior μ, and constants $\bar{C} \geq 1$ and $\hat{C} > 1$ such that expressions (8) and (9) hold simultaneously.

Set B: The beliefs are given below. The BDP property fails for agent 1.

$p_1(\theta_2 \mid \theta_1)$	θ_2^1	θ_2^2
θ_1^1	0.3	0.7
θ_1^2	0.3	0.7
θ_1^3	0.5	0.5

$p_2(\theta_1 \mid \theta_2)$	θ_2^1	θ_2^2
θ_1^1	0.2	0.3
θ_1^2	0.3	0.2
θ_1^3	0.5	0.5

The profile of beliefs satisfies Property I and Property II for agent 1. To see this, one may try conjecturing $\hat{\theta}_1 = \theta_1^1$, θ_1^2 or θ_1^3. However, in all cases, one can never find μ to satisfy expression (6) (or (8)).

Set C: Consider the profile of beliefs in Example 4.2. By relabeling agents 1 and 2 there, we can obtain a three-by-two example. We have verified that the BDP property holds for both agents. Also, the profile of beliefs satisfies Property I and Property II for both agents.

Set D: The beliefs are given below. The BDP property holds for agent 1.

$p_1(\theta_2 \mid \theta_1)$	θ_2^1	θ_2^2
θ_1^1	$\frac{3}{7}$	$\frac{4}{7}$
θ_1^2	$\frac{4}{7}$	$\frac{3}{7}$
θ_1^3	$\frac{5}{26}$	$\frac{21}{26}$

$p_2(\theta_1 \mid \theta_2)$	θ_2^1	θ_2^2
θ_1^1	0.1	0.2
θ_1^2	0.4	0.1
θ_1^3	0.5	0.7

$\mu(\theta)$	θ_2^1	θ_2^2
θ_1^1	$\frac{1}{40}$	$\frac{6}{40}$
θ_1^2	$\frac{4}{40}$	$\frac{3}{40}$
θ_1^3	$\frac{5}{40}$	$\frac{21}{40}$

The profile of beliefs does not satisfy Property I for agent 1, because there exist types $(\bar{\theta}_i, \hat{\theta}_i) = (\theta_1^2, \theta_1^1)$, a distribution μ defined above, and constants $\bar{C} = 0.5$ and $\hat{C} = 1.5$, such that expressions (6) and (7) hold.

The profile of beliefs satisfies Property II for agent 1. The argument is similar to the one for Set A.

Set E: The beliefs are given below. The BDP property holds for agent 1.

$p_1(\theta_2\|\theta_1)$	θ_2^1	θ_2^2
θ_1^1	0.7	0.3
θ_1^2	0.3	0.7
θ_1^3	$\frac{12}{19}$	$\frac{7}{19}$

$p_2(\theta_1\|\theta_2)$	θ_2^1	θ_2^2
θ_1^1	0.4	0.1
θ_1^2	0.1	0.4
θ_1^3	0.5	0.5

$\mu(\theta)$	θ_2^1	θ_2^2
θ_1^1	$\frac{48}{190}$	$\frac{7}{190}$
θ_1^2	$\frac{12}{190}$	$\frac{28}{190}$
θ_1^3	$\frac{60}{190}$	$\frac{35}{190}$

The profile of beliefs does not satisfy Property I or Property II for agent 1. To see this, consider types $(\bar{\theta}_i, \hat{\theta}_i) = (\theta_1^2, \theta_1^1)$, the distribution μ defined above, and constants $\bar{C} = \frac{32}{19}$ and $\hat{C} = \frac{63}{19}$. Then, expressions (6), (7), (8), and (9) hold simultaneously.

Set F: The beliefs are given below. The BDP property fails for agent 1.

$p_1(\theta_2\|\theta_1)$	θ_2^1	θ_2^2
θ_1^1	$\frac{3}{7}$	$\frac{4}{7}$
θ_1^2	$\frac{4}{7}$	$\frac{3}{7}$
θ_1^3	$\frac{4}{7}$	$\frac{3}{7}$

$p_2(\theta_1\|\theta_2)$	θ_2^1	θ_2^2
θ_1^1	$\frac{1}{9}$	$\frac{1}{5}$
θ_1^2	$\frac{4}{9}$	$\frac{2}{5}$
θ_1^3	$\frac{4}{9}$	$\frac{2}{5}$

$\mu(\theta)$	θ_2^1	θ_2^2
θ_1^1	$\frac{2}{33}$	$\frac{3}{33}$
θ_1^2	$\frac{8}{33}$	$\frac{6}{33}$
θ_1^3	$\frac{8}{33}$	$\frac{6}{33}$

The profile of beliefs does not satisfy Property I or Property II for agent 1. To see this, consider types $(\bar{\theta}_i, \hat{\theta}_i) = (\theta_1^2, \theta_1^1)$, the distribution μ defined above, and constants $\bar{C} = 14$ and $\hat{C} = 6$. Thus, expressions (6), (7), (8), and (9) hold.

References

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