Persuasion with Sequential Private Information

Huiyi Guo* Wei He†

May 29, 2024

Abstract

This paper studies a Bayesian persuasion problem where the agent learns his private information gradually. We find that the optimal persuasion mechanism may offer different experiments to different types of agents. In particular, the agent's late-stage precise private information is of no screening value to the principal, but the agent's early-stage rough information may be of screening value. This finding contrasts with the optimality of non-discriminatory information disclosure in some commonly studied scenarios with one stage of private information. We identify necessary and sufficient condition under which the optimal information disclosure is non-discriminatory. Our results demonstrate that time can be utilized as an instrument to achieve information discrimination.

*Huiyi Guo: Department of Economics, Texas A&M University, 4228 TAMU, College Station, TX 77843, United States of America. Email: huiyiguo@tamu.edu.

†Wei He: Department of Economics, The Chinese University of Hong Kong, Hong Kong, China. Email: hewei@cuhk.edu.hk.

Contents

1 Introduction

Examples are abundant in which a principal (she) releases information to persuade an agent (he). For instance, an online platform such as Pinterest or Instagram recommends content that showcases a product's quality, to entice the user to purchase through its "shop now" link; a project manager reveals information about the worthiness of a project, aiming to convince a worker to undertake this project; a search committee chair presents information about the research potential of a candidate, to persuade the department head to make an offer. Nevertheless, it is often the case that the agent gradually learns some additional private information, which, along with the information released by the principal, helps him to make a decision. For example, the Pinterest user may gradually refine his idiosyncratic taste for this product; the worker may gradually discover his private cost of working on the project; the department head may update his private assessment of the candidate's match quality with the department throughout time.

In this paper, we study the design of a persuasion mechanism so that the principal releases information about the state of the world to best influence the action of the agent. The agent learns his private information gradually. He first receives a rough private signal (type) in stage one. This rough private signal conveys noisy information of his refined private information to be realized in stage two, which can be interpreted as his private cost or bar of taking the principal's preferred action. Eventually, the agent bases his decision on the state of the world and his own refined private information, although the principal always prefers the agent to take one action.

In general, the principal can commit to a menu of experiments about the state of the world contingent on the agent's report of his stage-one rough private signal and stage-two refined private information. If the menu consists of at least two experiments, then we say the optimal persuasion mechanism entails information discrimination. Otherwise, the optimal persuasion mechanism is non-discriminatory.

We find that the optimal persuasion mechanism may or may not be discriminatory. In particular, when it is discriminatory, it must be implemented by a menu of experiments contingent on the agent's report of stage-one private information only. We provide a necessary and sufficient condition, as well as easy-to-check conditions, on when the optimum necessitates information discrimination. One such sufficient condition essentially implies that the high type dominates the low type strongly. When the high type dominates the low type in a weak way, for instance, when the agent's stage-one type is independent of the stage-two private cost, the optimum is implementable via a single experiment.

The observation that the agent's stage-one private information may be of screening value, but not the stage-two information, complements findings in the literature. In some commonly studied environments where the agent directly observes his precise cost, it is shown that the optimum entails no information discrimination. See, *e.g.*, Kolotilin et al. (2017) and Guo and Shmaya (2019), which are discussed in detail in the literature review. Our stage-two private information corresponds to their private cost and is also of no screening value. However, our new observation is that the rough private information may be of screening value, *i.e.*, time can be utilized as an instrument to achieve information discrimination.

In reality, discriminatory information provision is seen in applications. For example, online platforms can utilize a user's browsing history to personalize recommended content. These browsing histories convey noisy information about the user's "bar" for buying a certain product and can be manipulated by the user easily, *e.g.*, by logging in from a different account. Our results rationalize such a practice of personalizing content.

Literature review This paper joins the works on information design with a privately informed agent. The setup of the current paper is most related to Kolotilin et al. (2017), where the information controlled by the sender is independently distributed to the private information of the agent. In their baseline model, Kolotilin et al. (2017) show that every incentive compatible persuasion mechanism can be implemented by a single experiment. Hence, the optimal persuasion mechanism can also be implemented by an experiment, *i.e.*, there is no value in information discrimination. In Guo and Shmaya (2019), the information controlled by the sender is correlated with the agent's private information. It is no longer true that every incentive compatible persuasion mechanism can be implemented by a single experiment, but the optimal one can be implemented without information discrimination. The current paper complements the above papers by showing that when the agent's private information arrives gradually, even though these pieces of information are independent of the one controlled by the sender, not every incentive compatible persuasion mechanism can be implemented by a single experiment, including the optimal one. Hence, there is a value in information discrimination.

The assumption that the agent gradually learns his private information, which presents an opportunity for the principal to sequentially screen the agent's private information, has been seen in the literature on mechanism design as well as that on delegation. In the mechanism design literature, Courty and Li (2000) study the optimal selling mechanism for a good when the agent first learns a rough private valuation of a product, *i.e.*, his type, and then learns his precise private valuation. When only the interim participation constraint is imposed on the mechanism, they show that the optimal selling mechanism is a menu of option contracts, with different contracts designed for different types. When strengthening the interim participation constraint with the ex-post one, Krähmer and Strausz (2015) establish a decreasing cross-hazard rate condition under which the optimal selling mechanism is static, *i.e.*, offers the same contract to different types. This implies that there is no value in screening the agent's type. By relaxing the decreasing cross-hazard rate condition and keeping the ex-post participation constraint, Bergemann et al. (2020) further show that the optimal selling mechanism may or may not offer the same contract to different types and characterize when the optimal mechanism is static. In the literature on delegation, Krähmer and Kováč (2016) study a problem where the agent learns his private information gradually. They show that the optimal delegation may or may not offer different delegation sets to different types, and provide sufficient conditions for the optimal delegation to be static or sequential. The current paper studies an information design problem, thereby differing from the above-mentioned papers. However, as our optimal persuasion mechanism may feature a menu of experiments, with different experiments designed for different types, our results bear some similarity with the insights from these papers, especially those from Krähmer and Kováč (2016) and Bergemann et al. (2020).

The information design literature has discussed several factors that lead the optimal persuasion to involve information discrimination. Kolotilin et al. (2017) have discussed several such factors, including nonlinearity, larger action sets, etc. Another important scenario that benefits from information discrimination is when the problem involves joint design in information and pricing. See, for example, Li and Shi (2017), Guo et al. (2024), and Wei and Green (2024). Moreover, when there are multiple agents, the sender may leverage the heterogeneity among agents by discriminatorily offering information. See, *e.g.,* Bardhi and Guo (2018), Bobkova and Klein (2018), Chan et al. (2019). In the current paper, we point out a new factor that contributes to profitable information discrimination, *i.e.*, the gradual arrival of private information.

Finally, there is a rich strand of the literature embedding dynamics to information design. For instance, when multiple senders move sequentially, as in Li and Norman (2021) and Wu (2023), the problem naturally features sequential information design. Alternatively, the state of the world can evolve dynamically, as in Ely (2017). Last but not least, it can be the case that information production takes time, as in Che et al. (2023). In the current paper, the source of dynamics is the evolution of agent's private information, which differentiates the current paper from most other papers in this strand of the literature.

2 Setup

Consider a relationship between a principal (she) and an agent (he). There are two actions that the agent can take from $A = \{0, 1\}$. Action $a = 0$ generates no benefit or cost to the principal and the agent. Action $a = 1$ leads to a random benefit $\omega \in \Omega = [0, 1]$ and incurs a private cost c to the agent, yet it always benefits the principal. Let the principal's and the agent's utilities be

$$v(\omega, c, a) = a \quad \text{and} \quad u(\omega, c, a) = a(\omega - c).$$

In stage one, the agent learns his type $t \in \{L, H\}$, which is realized with probability $f(t)$ and conveys noisy information about his private cost of taking action $a = 1$. The agent's private cost, $c \in C \equiv [0, 1]$, is realized in stage two. Given that t is learned in stage one, the CDF and PDF of c are given by $G_t(\cdot)$ and $g_t(\cdot)$. Let $g(\cdot) = f(L)g_L(\cdot) + f(H)g_H(\cdot)$ be the unconditional density of c. We impose the following assumptions throughout the paper.

Assumption 1. *PDFs $g_L(c)$, $g_H(c)$, and $g(c)$ are positive, continuously differentiable, and log-concave in c.*

Assumption 2. *The likelihood ratio* $\dfrac{g_H(c)}{g_L(c)}$ *is strictly increasing in* c.

We may also call the random benefit ω the state, which is realized only at stage three. The state ω follows CDF Φ which has continuous and fully supported PDF ϕ. Moreover, ω is assumed to be independent of (t, c). The principal controls the releasing of information about ω.

The principal designs a persuasion mechanism in stage zero to maximize her expected payoff. A persuasion mechanism $\pi : \Omega \times T \times C \to [0, 1]$ requires the agent to sequentially reveal t and c, and then recommends the agent to take action $a = 1$ (resp. $a = 0$) with probability $\pi(\omega, t, c)$ (resp. $1 - \pi(\omega, t, c)$).

The agent, after receiving the recommendation from the principal, can choose to follow it or not. Let a_1 (resp. a_0) be the action taken by the agent when he receives recommendation $a = 1$ (resp. $a = 0$). Let $\hat{t}$ and $\hat{c}$ be the agent's reported private information in stage one and stage two. Then, the agent with true information t and c has the following expected payoff in stage two:

$$U_\pi(\hat{t}, \hat{c}, a_0, a_1 | t, c) \equiv \int_\Omega \Big[a_0(1 - \pi(\omega, \hat{t}, \hat{c})) + a_1 \pi(\omega, \hat{t}, \hat{c}) \Big] (\omega - c) \, \mathrm{d}\Phi(\omega).$$

The persuasion mechanism should give the agent incentive to truthfully report in both stages and follow the recommendation. We decompose this requirement into two conditions: incentive compatibility conditions in stage two (IC_2) and one (IC_1).

The IC_2 condition requires that the agent has no incentive to engage in one-shot deviation in stage two, which includes misreporting c and disobeying the recommendation sent by the principal. Formally, IC_2 requires that $U_\pi(t, c) \equiv U_\pi(t, c, 0, 1|t, c) \geqslant U_\pi(t, \hat{c}, a_0, a_1|t, c)$, *i.e.*, for all $t \in T$, $c \in C$, $\hat{c} \in C$, and $a_0, a_1 \in A$,

$$\int_\Omega \pi(\omega, t, c)(\omega - c) \, \mathrm{d}\Phi(\omega) \geqslant \int_\Omega \Big[a_0(1 - \pi(\omega, t, \hat{c})) + a_1 \pi(\omega, t, \hat{c}) \Big] (\omega - c) \, \mathrm{d}\Phi(\omega). \tag{1}$$

The IC_1 condition requires that the agent has no incentive to deviate starting from stage one, which includes one-shot deviation in stage one and double deviation in both stages. Formally, IC_1 requires that

$$\int_{c \in C} U_\pi(t, c) g_t(c) \, \mathrm{d}c \geqslant \int_{c \in C} U_\pi(\hat{t}, \mathbf{c}(c), \mathbf{a}_0(c), \mathbf{a}_1(c) | t, c) g_t(c) \, \mathrm{d}c$$

for all $t, \hat{t} \in T$, stage-two reporting plan $\mathbf{c} : C \to C$, action plan contingent on the principal's recommendation $a = 0$, *i.e.*, $\mathbf{a}_0 : C \to A$, and action plan contingent on the principal's recommendation $a = 1$, *i.e.*, $\mathbf{a}_1 : C \to A$.

The principal's original problem, **Problem (O)**, is defined as follows:

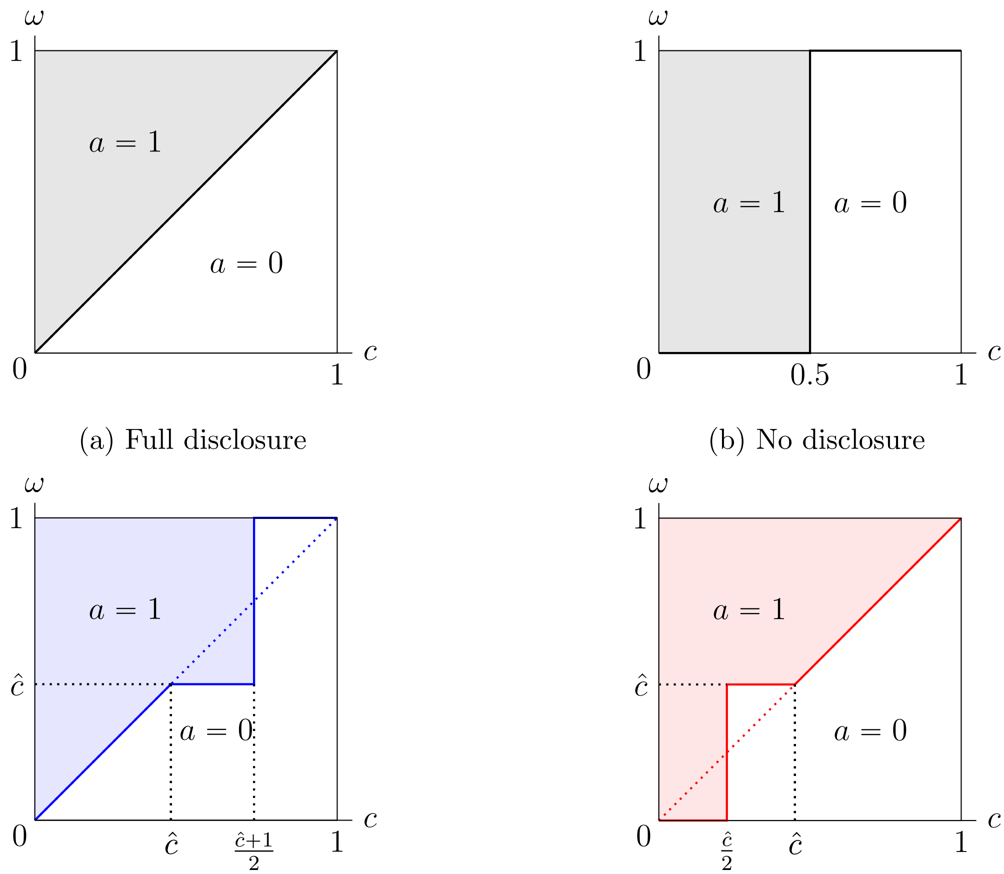
$$\begin{aligned} \max_{\pi} \quad & \int_{t \in T} \int_{c \in C} \int_{\omega \in \Omega} \pi(\omega, t, c) \phi(\omega) g_t(c) f(t) \, \mathrm{d}\omega \, \mathrm{d}c \, \mathrm{d}t \\ \text{s.t.} \quad & \pi \text{ satisfies } IC_1 \text{ and } IC_2. \end{aligned}$$

A persuasion mechanism discloses information about the state by sending a recommendation after eliciting the agent's report of his private information in two stages. Instead of adopting a persuasion mechanism, the principal can also disclose information about the state by using an experiment, which does not require the agent to report private information. An (direct) experiment is a mapping $\sigma : \Omega \to M \equiv [0, 1]$ where $m = \mathbb{E}[\omega | \sigma(\omega) = m]$. Namely, the experiment sends a piece of message $m \in M$ contingent on the state, such that the message directly informs the agent of the posterior mean state that sends m. An important question is whether a persuasion mechanism can be implemented by an experiment or not, *i.e.*, whether there exists an experiment that provides the same expected utility to the principal as the persuasion mechanism. If there is such an experiment, then we say the persuasion mechanism is non-discriminatory or entails no information discrimination; otherwise, the persuasion mechanism is discriminatory, or entails information discrimination.

3 Example

Suppose the state ω, type t, and private cost c are all uniformly distributed, *i.e.*, $\phi(\omega) = 1$ for all $\omega \in \Omega$, $f(t) = 0.5$ for both $t \in T$, and $g(c) = 1$ for all $c \in C$. The realization of ω is independent of t and c, but t and c are correlated with conditional PDFs being $g_L(c) = 2-2c$ and $g_H(c) = 2c$ for all $c \in C$.

Suppose the principal commits to offering an experiment to the agent. Figure 1 showcases four simple experiments and the induced actions: the experiments in panels (a) and (b) involve fully disclosing ω and not disclosing ω respectively; the experiment in panel (c) involves upper censorship with cutoff $\hat{c}$, *i.e.*, fully disclosing $\omega \in [0, \hat{c}]$ and pooling states



(a) Full disclosure

(b) No disclosure

(c) Upper censorship

(d) Lower censorship

Figure 1: Four experiments

$\omega \in (\hat{c}, 1]$; the experiment in panel (d) involves lower censorship with cutoff $\hat{c}$, *i.e.*, pooling states $\omega \in [0, \hat{c}]$ and fully disclosing $\omega \in (\hat{c}, 1]$. The shaded area in each panel describes the range of ω and c where the agent takes action $a = 1$. As ω and c are uniformly and independently distributed, the pair (ω, c) falls into the shaded area with 50 percent probability in each panel, implying that the principal earns an ex ante payoff of 0.5 under each of the four experiments. In fact, not only these four experiments are equally good: by applying the argument of Kolotilin et al. (2017) to the current example, it is easy to show that every experiment leads to the same ex ante payoff to the principal, which is equal to 0.5. Moreover, their result implies that the principal's ex ante payoff payoff remains 0.5 if she is allowed to screen c, *i.e.*, to provide different experiments to an agent with different private costs.

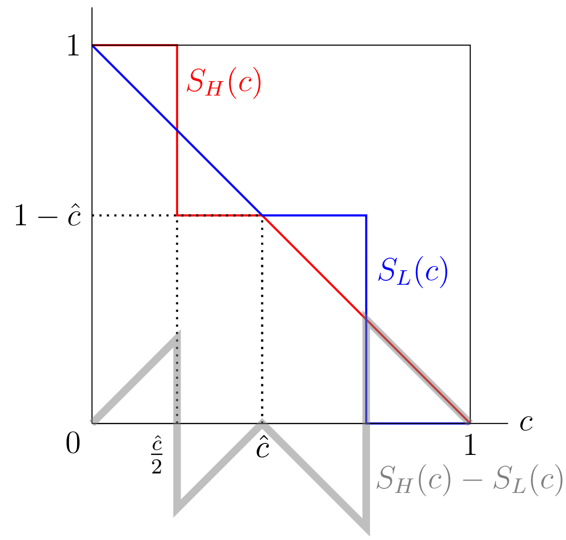


Figure 2: Surplus extracted from cost-c agent

To see that information discrimination is profitable, we begin with the upper censorship experiment with cutoff $\hat{c}$, where $\hat{c}$ is slightly below 0.5.

Then consider a menu of two experiments: the one designed for type L is the upper censorship experiment with cutoff $\hat{c}$ and the one designed for type H is the lower censorship one with the same cutoff. Function $S_L(c)$ (resp. $S_H(c)$) in Figure 2 shows the surplus the principal can extract from an agent with cost c under the experiment designed for type L (resp. H), and this function is derived from the height of the shaded area in panel (c) (resp. (d)) of Figure 1.

By switching from the non-discriminatory persuasion to the discriminatory one which entails a menu of experiments, the principal extracts the same surplus from type L. The additional surplus extracted from type-H cost-c agent is given by the function $S_H(c) - S_L(c)$ in Figure 2, which has positive expectation under PDF $g_H(c)$ given $\hat{c} < 0.5$. As a result, the discriminatory persuasion is profitable, if each type chooses his designated experiment.

The upper and lower censorship experiments, labeled by L and H respectively in the following expressions, give cost-c agent expected payoffs of

$$U_L(c) = \begin{cases} \int_c^1 (\omega - c)\,\mathrm{d}\omega, & c \in [0, \hat{c}]; \\ (1-\hat{c})(\frac{\hat{c}+1}{2} - c), & c \in (\hat{c}, \frac{\hat{c}+1}{2}]; \\ 0, & c \in (\frac{\hat{c}+1}{2}, 1]; \end{cases} \quad U_H(c) = \begin{cases} 0.5 - c, & c \in [0, \frac{\hat{c}}{2}]; \\ (1-\hat{c})(\frac{\hat{c}+1}{2} - c), & c \in (\frac{\hat{c}}{2}, \hat{c}]; \\ \int_c^1 (\omega - c)\,\mathrm{d}\omega, & c \in (\hat{c}, 1]. \end{cases}$$

In particular, the $U_L(c)$ function single crosses the $U_H(c)$ function for one time and from above when c increases from 0 to 1. It is easy to verify numerically that when $\hat{c}$ is only slightly below 0.5, say $\hat{c} = 0.475$, PDF g_L (resp. g_H) evaluates U_L (resp. U_H) more favorably. Hence, the menu of experiments is incentive compatible, *i.e.*, each type at least weakly prefers his designated experiment.

Finally, we remark that the above construction may not work under other PDFs. For instance, we focus on $\hat{c} = 0.475$ and replace the conditional PDFs by $\hat{g}_L(c) = \frac{1}{2} + \ln 2 - \frac{1}{2}\ln(c+1)$, which is convex and decreasing, and $\hat{g}_H(c) = \frac{3}{2} - \ln 2 + \frac{1}{2}\ln(c+1)$, which is concave and increasing. These two PDFs are also flatter than g_L and g_H. We provide a heuristic argument to see why the previous construction does not work under $\hat{g}_L$ and $\hat{g}_H$. With the change in PDFs, it is more difficult for $S_H - S_L$ to have a positive expectation under PDF $\hat{g}_H$. This is because that concave PDF $\hat{g}_H$ overall puts relatively more weight on the negative middle portion of $S_H - S_L$ than the linear and increasing g_H does. Meanwhile, it is more difficult for the L type to prefer the experiment that is desigated for her. This is because the flatter $\hat{g}_L$ overall puts relatively more (resp. less) weight on the negative (resp. positive) portion of $U_L - U_H$ than g_L. In fact, as we will formally show in Section 4, under PDFs $\hat{g}_L$ and $\hat{g}_H$, there does not exist any menu of experiments that is strictly more profitable than the optimal single experiment.

4 Analysis and results

4.1 Equivalent problem

Define $\overline{U}(c) = \int_c^1 (\omega - c)\,\mathrm{d}\Phi(\omega)$ and $\underline{U}(c) = \max\{\mathbb{E}[\omega] - c, 0\}$, which are cost-$c$ agent's expected payoffs under full and no disclosure, respectively. Let $\mathcal{U}$ be the set of *convex* functions $U : C \to \mathbb{R}$ bounded above by $\overline{U}$ and below by $\underline{U}$.

We now define another problem, where the principal does not directly choose a persuasion mechanism. Her choice variables are the two types' indirect utility functions $U_L, U_H \in \mathcal{U}$, where type-t cost-c agent earns expected utility $U_t(c)$ from the underlying persuasion mechanism. This is **Problem (I)**:

$$\begin{aligned}
\max_{U_L, U_H \in \mathcal{U}} \quad & \left[g(0)\mathbb{E}[\omega] + f(L) \int_{c \in C} U_L(c) g_L'(c) \, \mathrm{d}c + f(H) \int_{c \in C} U_H(c) g_H'(c) \, \mathrm{d}c \right] \\
\text{s.t.} \quad & \int_{c \in C} U_L(c) g_L(c) \, \mathrm{d}c \geqslant \int_{c \in C} U_H(c) g_L(c) \, \mathrm{d}c, \\
& \int_{c \in C} U_H(c) g_H(c) \, \mathrm{d}c \geqslant \int_{c \in C} U_L(c) g_H(c) \, \mathrm{d}c.
\end{aligned}$$

Note that $g(0)\mathbb{E}[\omega]$ is a constant number and we can omit it from the maximization problem.

Lemma 1 shows that this newly defined problem is equivalent to the original problem that the principal has.

Lemma 1. *(a) For every persuasion mechanism π that satisfies the constraints in Problem (O), the induced indirect utility functions for the two types satisfy all constraints in Problem (I) and lead to the same value in Problem (I) as π does in Problem (O).*

(b) For every pair of $U_L, U_H \in \mathcal{U}$ that satisfies the constraints in Problem (I), there exists a persuasion mechanism π that satisfies all constraints in Problem (O) and leads to the same value in Problem (O) as the pair of $U_L, U_H \in \mathcal{U}$ does in Problem (I).

4.2 Non-discriminatory persuasion benchmark

To study when it is optimal to adopt non-discriminatory or discriminatory persuasion, the solution of the optimal non-discriminatory persuasion turns out to play an important role. Hence, we identify a solution of the optimal non-discriminatory persuasion in this section.

If the principal does not elicit the agent's stage-one private information, then the principal's problem can be written as **Problem (S)** (where S refers to static) defined as follows:

$$\max_{U \in \mathcal{U}} \int_{c \in C} U(c) g'(c) \, \mathrm{d}c.$$

For each $s \in [0,1]$, we define an experiment $\sigma_{(0,s)}$ as follows: let the experiment fully reveal $\omega \in [0,s]$ and pool $\omega \in (s,1]$ into the posterior mean $\mathbb{E}[\omega|\omega > s]$.[1] We call this experiment an **upper censorship experiment** and denote the induced expected utility to cost-c agent by $U_{(0,s)}(c)$. By Assumption 1, $\frac{g'}{g}$ is decreasing. Hence, g' satisfies the **downward single crossing property** (DSCP), *i.e.*, there do not exist $s_1, s_2 \in C$ with $s_1 < s_2$ such that $g'(s_1) < 0$ and $g'(s_2) > 0$. The additive inverse of a function satisfying the DSCP is said to satisfy the **upward single crossing property** (USCP). Given g' satisfying the DSCP, we identify one of its crossing points with the horizontal axis as follows:

$$\hat{s} \equiv \begin{cases} \inf\{s \in C : g'(s) < 0\}, & \text{if } \{s \in C : g'(s) < 0\} \neq \varnothing; \\ 1, & \text{if } \{s \in C : g'(s) < 0\} = \varnothing. \end{cases} \tag{2}$$

When g' satisfies Assumption 1, and thus, DSCP, Kolotilin et al. (2017) show that the solution of Problem (S) can be implemented by an upper censorship experiment.

We define a candidate cutoff for an upper censorship experiment, s^*, as follows:

$$s^* \equiv \min\{s \in [0,\hat{s}] : s \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s], \int_s^{\mathbb{E}[\omega|\omega>s]} (c-s)g'(c)\,\mathrm{d}c \leqslant 0\}. \tag{3}$$

The last requirement in the brace is the first order condition, where the weak inequality takes care of the potential corner solution $s^* = 0$. Notice that the other potential corner solution $s^* = 1$ happens only if $s^* = \hat{s} = \mathbb{E}[\omega|\omega > s^*] = 1$, for which the first order condition holds as an equality. The requirement that $s \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s]$ is essentially a second order condition given g' satisfies the DSCP. When there are multiple s^* satisfying the above two requirements, the minimization operator selects one s^*.

It is worth pointing out that, if $s^* > 0$, it must be the case that $\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g'(c)\,\mathrm{d}c = 0$.[2] Equivalently,

$$f(L)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_L'(c)\,\mathrm{d}c + f(H)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_H'(c)\,\mathrm{d}c = 0. \tag{4}$$

[1] As ϕ, g_H, and g_L are continuous, this experiment induces the same expected payoff to the principal and to any cost-c agent as one that fully reveals $\omega \in [0,s)$ and pools $\omega \in [s,1]$. Hence, when $s = 1$, we abuse notation by letting $\mathbb{E}[\omega|\omega > 1] = 1$.

[2] To see this, if $s^* > 0$ and $\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g'(c)\,\mathrm{d}c < 0$, it must be the case that $s^* \leqslant \hat{s} < \mathbb{E}[\omega|\omega > s^*]$. Then there exist $s \in (0,s^*) \subseteq [0,\hat{s}]$ such that $\int_s^{\mathbb{E}[\omega|\omega>s]} (c-s)g'(c)\,\mathrm{d}c < 0$ and $s < \hat{s} < \mathbb{E}[\omega|\omega > s]$. This contradicts with the definition of s^*.

The following lemma verifies that $U_{(0,s^*)}$ is a solution of Problem (S).

Lemma 2. *Given the above defined s^*, $U_{(0,s^*)}$, the indirect utility function induced by the upper censorship experiment $\sigma_{(0,s^*)}$, solves Problem (S).*

4.3 Necessary and sufficient condition

We now provide a necessary and sufficient condition on when non-discriminatory/discriminatory disclosure is optimal. Moreover, we describe a feature of the experiment or the menu of experiments that implements the optimal persuasion mechanism.

Proposition 1. *(a) The optimal persuasion mechanism can be implemented by an experiment if and only if for all $c_1 < s^*$ and $c_2 > s^*$,*

$$\frac{g'_H(c_1)}{g_L(c_1)} \geqslant -\frac{f(L)}{f(H)} \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g'_L(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c} \geqslant \frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g'_H(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g_L(c)\,\mathrm{d}c}. \tag{5}$$

(b) When condition (5) is satisfied, the optimal persuasion mechanism can be implemented by $\sigma_{(0,s^*)}$; otherwise, in the menu of experiments that implements the optimal persuasion mechanism,

- *there exist $c_L \in C$ such that the experiment designed for L type is $\sigma_{(0,c_L)}$;*
- *the L-type agent is indifferent between the two experiments.*

When $s^* > 0$, due to (4), the key condition in the above proposition, (5), can be equivalently be written as

$$\frac{g'_H(c_1)}{g_L(c_1)} \geqslant \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g'_H(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c} \geqslant \frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g'_H(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g_L(c)\,\mathrm{d}c}. \tag{6}$$

This condition relies directly on g'_H and g_L as well as the endogenously solved s^*. The first term in the inequality is the ratio of g'_H and g_L evaluated at $c_1 < s^*$, the second term can be interpreted as the ratio of a weighted sum of g'_H over a weighted sum of g_L in the range $[s^*, \mathbb{E}[\omega|\omega > s^*]]$, and the last term is also a ratio of weighted sums (with weights different from before) of g'_H and g_L in a range with either a higher upper limit of integration (if $c_2 > \mathbb{E}[\omega|\omega > s^*]$) or a higher lower limit of integration (if $s^* < c_2 \leqslant \mathbb{E}[\omega|\omega > s^*]$).

To establish the only if direction of part (a), we proceed by discussing two cases and describing the construction of a menu of experiments illustrated in Figure 3.

Case I: Suppose the LHS inequality in (5) is violated for some $c_1 < s^*$. Obviously, $s^* > 0$ in this case. Let a new experiment designed for the L type be $\sigma_{(0,s^*-\epsilon)}$. Construct a new experiment for H type as follows:

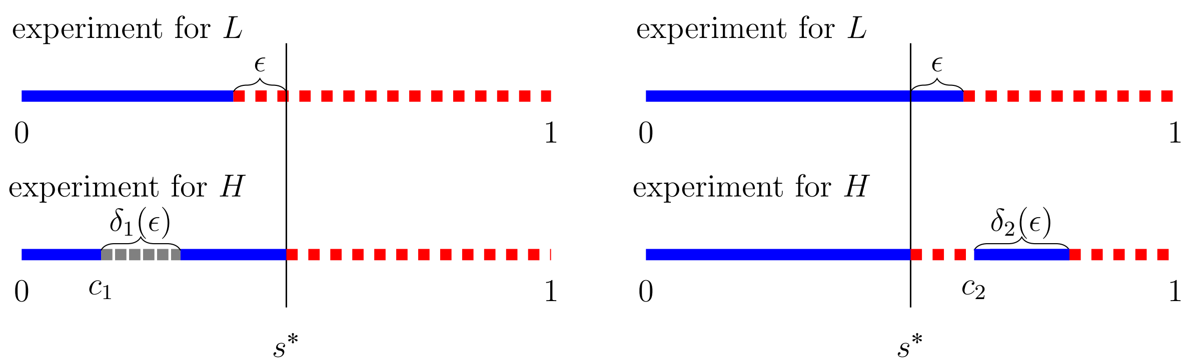
- fully revealing on $[0, c_1] \cup [c_1 + \delta_1(\epsilon), s^*]$;
- pooling on $(c_1, c_1 + \delta_1(\epsilon))$;
- pooling on $(s^*, 1]$.

The choice of $\delta_1(\epsilon)$ is such that the L type is indifferent between the experiments designed for the L type and the H type. We remark that $\epsilon > 0$ should be sufficiently small such that $s^* - \epsilon > 0$ and $c_1 + \delta_1(\epsilon) < s^*$.

Case II: Suppose the RHS inequality in (5) is violated for some $c_2 > s^*$. In this case, $s^* < 1$. Let the new experiment designed for type L be $\sigma_{(0,s^*+\epsilon)}$. Construct a new experiment for H type as follows:

- fully revealing on $[0, s^*] \cup [c_2, c_2 + \delta_2(\epsilon)]$;
- pooling on $(s^*, c_2) \cup (c_2 + \delta_2(\epsilon), 1]$.

Again, the choice of $\delta_2(\epsilon)$ is such that the L type is indifferent between the experiments designed for the L type and the H type. Also, ϵ should be sufficiently small such that $s^* + \epsilon < 1$ and $c_2 + \delta_2(\epsilon) < 1$.



(a) LHS inequality in (5) is violated

(b) RHS inequality in (5) is violated

Figure 3: Construction of a menu of experiments: reveal ω in the blue solid area, pool the red loosely dotted area into one message, pool the gray densely dotted area into one message.

The formal analysis of why it is profitable to adopt this menu of experiments (which give the cost-c agent expected payoffs of $U_L(c)$ and $U_H(c)$ respectively) is relegated to the Appendix. Now, we utilize the group of linear PDFs g_L and g_H in Section 3, for which $s^* = 0$ and the RHS of condition (5) (and (6), too) is violated for every $c_2 > s^* = 0$, to gain some intuition.

Suppose $g_L(c) = 2 - 2c$ and $g_H(c) = 2c$, for which $\frac{g'_H(c)}{g_L(c)}$ is increasing. The binding upward IC_1 constraint can be rewritten as

$$\int_{c\in C} [U_H(c) - U_L(c)]g_L(c)\,\mathrm{d}c = 0,$$

where $U_H - U_L$ can be shown to satisfy the USCP. This equation as well as the USCP implies the downward IC_1 constraint

$$\int_{c\in C} [U_H(c) - U_L(c)]g_H(c)\,\mathrm{d}c = \int_{c\in C} [U_H(c) - U_L(c)]g_L(c)\frac{g_H(c)}{g_L(c)}\,\mathrm{d}c \geqslant 0.$$

Intuitively, the inequality holds because under Assumption 2, the relative weight imposed on the positive (negative) part of $U_H(c) - U_L(c)$ by g_H is overall bigger (smaller) than it is when the weight is g_L. The change in the principal's profit by adopting this menu is:

$$g(L)\int_{c\in C} [U_L(c) - U_{(0,s^*)}(c)]g'_L(c)\,\mathrm{d}c + g(H)\int_{c\in C} \underbrace{[U_H(c) - U_{(0,s^*)}(c)]}_{U_L - U_{(0,s^*)} + U_H - U_L} g'_H(c)\,\mathrm{d}c$$

$$= \int_{c\in C} [U_L(c) - U_{(0,s^*)}(c)] \underbrace{[g(L)g'_L(c) + g(H)g'_H(c)]}_{=0 \text{ for the linear PDFs in Section 3}} \mathrm{d}c + g(H)\int_{c\in C} [U_H(c) - U_L(c)]g'_H(c)\,\mathrm{d}c$$

$$=g(H)\int_{c\in C}[U_H(c)-U_L(c)]g_L(c)\frac{g_H'(c)}{g_L(c)}\,\mathrm{d}c.$$

Since $\frac{g_H'(c)}{g_L(c)}$ is strictly increaing, the relative weight imposed by g_H' on the positive (negative) part of $U_H(c)-U_L(c)$ is overall bigger (smaller) than it is when the weight is g_L. Hence, the change in profit is positive, *i.e.*, adopting the menu of experiments is profitable.

Notice that the above argument does not apply to the group of nonlinear PDFs in Section 3 ($\hat{g}_L$ and $\hat{g}_H$) which satisfies condition (5). To see this, $\frac{\hat{g}_H'(c)}{\hat{g}_L(c)}$ is decreasing, and thus, the relative weight imposed by $\hat{g}_H'$ on the positive (negative) part of $U_H(c)-U_L(c)$ is overall smaller (bigger) than it is when the weight is $\hat{g}_L$. Hence, the change in profit is negative.

The following relaxed variant of Problem (I), **Problem (II)**, with the upward IC_1 constraint, turns out to be useful in establishing Proposition 1:

$$\max_{U_L,U_H\in\mathcal{U}}\quad \left[f(L)\int_{c\in C}U_L(c)g_L'(c)\,\mathrm{d}c+f(H)\int_{c\in C}U_H(c)g_H'(c)\,\mathrm{d}c\right]$$
$$\text{s.t.}\quad \int_{c\in C}U_L(c)g_L(c)\,\mathrm{d}c\geqslant\int_{c\in C}U_H(c)g_L(c)\,\mathrm{d}c.$$

The Lagrangian of Problem (II) is given by

$$\begin{aligned}\mathcal{L}=&f(L)\int_{c\in C}U_L(c)g_L'(c)\,\mathrm{d}c+f(H)\int_{c\in C}U_H(c)g_H'(c)\,\mathrm{d}c+\lambda\int_{c\in C}\left[U_L(c)-U_H(c)\right]g_L(c)\,\mathrm{d}c\\=&\int_{c\in C}U_L(c)\left[f(L)g_L'(c)+\lambda g_L(c)\right]\mathrm{d}c+\int_{c\in C}U_H(c)\left[f(H)g_H'(c)-\lambda g_L(c)\right]\mathrm{d}c\\=&\int_{c\in C}U_L(c)\left[f(L)\frac{g_L'(c)}{g_L(c)}+\lambda\right]g_L(c)\,\mathrm{d}c+\int_{c\in C}U_H(c)\left[f(H)\frac{g_H'(c)}{g_L(c)}-\lambda\right]g_L(c)\,\mathrm{d}c.\end{aligned}$$

Let

$$\mathcal{L}_1=\int_{c\in C}U_L(c)\left[f(L)\frac{g_L'(c)}{g_L(c)}+\lambda\right]g_L(c)\,\mathrm{d}c,$$

and

$$\mathcal{L}_2=\int_{c\in C}U_H(c)\left[f(H)\frac{g_H'(c)}{g_L(c)}-\lambda\right]g_L(c)\,\mathrm{d}c.$$

To establish the if direction in Part (a) of Proposition 1, we assume that condition (5) is satisfied. Then it sufficies to show that $U_L=U_H=U_{(0,s^*)}$ solves Problem (II), because the omitted downward IC_1 would be trivially satisfied. To show that $U_L=U_H=U_{(0,s^*)}$ solves Problem (II), we first identify the candidate Lagrangian multiplier λ. Then, we show that with this λ, the optimal U_L for $\mathcal{L}_1$ is $U_L=U_{(0,s^*)}$ and the optimal U_H for $\mathcal{L}_2$ is $U_H=U_{(0,s^*)}$. Hence, the solution of Problem (II) can be implemented by the experiment $\sigma_{(0,s^*)}$.

To establish Part (*b*) of Proposition 1, recall the observation that $\sigma_{(0,s^*)}$ is the optimal single experiment from Lemma 2. Hence, we focus on the structure of the optimal menu. In Step 1, we show that there exists $c_L \in C$ such that the experiment designed for L type is $\sigma_{(0,c_L)}$. To achieve this, we begin with any pair of (U_L, U_H) with $U_L, U_H \in \mathcal{U}$ satisfying the constraint in Problem (II). We then construct certain $\hat{U}_L \in \mathcal{U}$ that is implementable by an upper censorship experiment so that $(\hat{U}_L, U_H)$ satisfies the constraint in Problem (II) and improves the principal's payoff relative to (U_L, U_H). In Step 2, we argue that the upward IC_1 constraint in the solution of Problem (II) must bind. In Step 3, we employ Assumption 2 as well as the structure of the menu of experiments to show that the omitted downward IC_1 constraint in Problem (I) is satisfied by the solution of Problem (II). Hence, the menu of experiments implements the solution of Problem (I).

4.4 Easy-to-check conditions

We now give condition (5) a closer look. It directly depends on the conditional PDFs of c, and indirectly depends on the PDFs of ω and t through the endogenously solved s^*.

We now introduce some easy-to-check sufficient conditions for condition (5) and negation of it without relying on the integral expresssions therein.

Corollary 1. *(a) When* $\dfrac{g'_H(c)}{g_L(c)}$ *is decreasing in* $c \in C$*, the optimal persuasion mechanism can be implemented by one experiment.*

(b) When (4) holds and $\dfrac{g'_H(c)}{g_L(c)}$ *is strictly increasing in* $c \in \big[0, \mathbb{E}[\omega|\omega > s^*]\big]$*, the optimal persuasion mechanism cannot be implemented by one experiment.*

We call the term $\dfrac{g'_H}{g_L}$ the cross hazard rate. Recall that $\dfrac{g'_H}{g_L} = \dfrac{g'_H}{g_H} \cdot \dfrac{g_H}{g_L}$. According to our Assumptions 1 and 2, $\dfrac{g'_H}{g_H}$ is decreasing and $\dfrac{g_H}{g_L}$ is strictly increasing. Hence, the property of the cross hazard rate depends on the dominant force between the competing two. In the limiting case where $g_H \to g_L$, i.e., stage-one type is close to independent of $c \in C$, the cross hazard rate is decreasing, and accordingly, the optimal persuasion entails no information discrimination. Put differently, the H type has to dominate the L type in a strong way, in order for screening to be valuable.

We now restrict attention to monotone g_L and g_H. If g_H and g_L are both increasing or decreasing, and g_H is concave, then the cross hazard rate is decreasing; if g_H is strictly increasing and convex, and g_L is strictly decreasing, then the cross hazard rate is strictly increasing. The PDFs g_L and g_H in Section 3 fit into the latter case, although $\frac{\hat{g}'_H}{\hat{g}_L}$ therein is decreasing.

To see why (4) is imposed in Part (*b*), we modify only one PDF in the group of linear PDFs in Section 3 — let $g_H(c) = 0.5 + c$. In this case, the cross hazard rate is still strictly increasing, $s^* = 0$, and $\int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g'(c)\, \mathrm{d}c < 0$. Yet, (5) is satisfied, meaning that the optimal persuasion mechanism can be implemented by one experiment.

For non-monotone g_L and g_H, we present another example to show that the optimal persuasion may or may not involve information discrimination.

Example 1. *We now consider two log-concave PDFs in the beta distribution family:* $g_L(c) = \lambda c^{\alpha-1}(1-c)^{\beta-1}$ *and* $g_H(c) = \lambda c^{\beta-1}(1-c)^{\alpha-1}$ *with* $1 < \alpha < \beta$*, where* λ *is such that these functions are well-defined PDFs. Given the current parameter restrictions,* g_L *and* g_H *are single-peaked at* $\frac{\alpha-1}{\alpha+\beta-2}$ *and* $\frac{\beta-1}{\alpha+\beta-2}$*, respectively. Moreover,* ω *and* t *are both assumed to be uniform distributions. It is easy to verify that Assumption 2 is satisfied.*

For $\alpha = 2.7$*,* $\beta = 3.3$*,* g *is log-concave, and the cross hazard rate is decreasing. In this case, the optimal persuasion mechanism can be implemented by one experiment.*

For $\alpha = 2$*,* $\beta = 4$*,* g *is log-concave, and the peak of the cross hazard rate is reached when* $c = 0.65$*. One can also derive that* $s^* = 0.27$ *and that* $\mathbb{E}[\omega|\omega > s^*] < 0.65$*. In this case, the cross hazard rate is strictly increasing in the interval* $\big[0, \mathbb{E}[\omega|\omega > s^*]\big]$*. Hence, the optimal persuasion mechanism involves information discrimination.*

Notice that under the second of group parameters, the two PDFs are more "distant" from each other compared to those in the first group.

5 Concluding remarks

This paper studies a persuasion problem where the agent gradually learns his private information in two stages. In stage one, the agent observes a binary type, which conveys noisy information about his private cost, and in stage two, the agent perfectly observes the private cost. We show that the optimal persuasion mechanism may or may not involve information discrimination and provide necessary and sufficient conditions on when the optimum necessitates information discrimination.

A natural extension of the model is to go beyond the binary-type case. In fact, it is possible to follow the argument of the current paper to provide conditions under which information discrimination is or is not profitable. We provide the details in Online Appendix B. The general structure of the optimal persuasion mechanism remains an open question.

References

Bardhi, A. and Guo, Y. (2018). Modes of persuasion toward unanimous consent. *Theoretical Economics*, 13(3):1111–1149.

Bergemann, D., Castro, F., and Weintraub, G. Y. (2020). The scope of sequential screening with ex post participation constraints. *Journal of Economic Theory*, 188:105055.

Bobkova, N. and Klein, S. (2018). Persuading an informed committee. *Working paper*.

Chan, J., Gupta, S., Li, F., and Wang, Y. (2019). Pivotal persuasion. *Journal of Economic theory*, 180:178–202.

Che, Y.-K., Kim, K., and Mierendorff, K. (2023). Keeping the listener engaged: a dynamic model of bayesian persuasion. *Journal of Political Economy*, 131(7):1797–1844.

Courty, P. and Li, H. (2000). Sequential screening. *The Review of Economic Studies*, 67(4):697–717.

Ely, J. C. (2017). Beeps. *American Economic Review*, 107(1):31–53.

Guo, Y., Li, H., and Shi, X. (2024). Optimal discriminatory disclosure. Working Paper.

Guo, Y. and Shmaya, E. (2019). The interval structure of optimal disclosure. *Econometrica*, 87(2):653–675.

Kolotilin, A., Mylovanov, T., Zapechelnyuk, A., and Li, M. (2017). Persuasion of a privately informed receiver. *Econometrica*, 85(6):1949–1964.

Krähmer, D. and Kováč, E. (2016). Optimal sequential delegation. *Journal of Economic Theory*, 163:849–888.

Krähmer, D. and Strausz, R. (2015). Optimal sales contracts with withdrawal rights. *The Review of Economic Studies*, 82(2):762–790.

Li, F. and Norman, P. (2021). Sequential persuasion. *Theoretical Economics*, 16(2):639–675.

Li, H. and Shi, X. (2017). Discriminatory information disclosure. *American Economic Review*, 107(11):3363–3385.

Wei, D. and Green, B. (2024). (reverse) price discrimination with information design. *American Economic Journal: Microeconomics*, 16(2):267–295.

Wu, W. (2023). Sequential bayesian persuasion. *Journal of Economic Theory*, 214:105763.

A Appendix A: Proofs

A.1 Proof of Lemma 1

We establish this lemma by showing that Problem (O) and Problem (I) have equivalent constraints and objective functions.

Step 1: Establish the equivalence between IC_2 and $U_L, U_H \in \mathcal{U}$.

The following result, directly implied by Theorem 1 of Kolotilin et al. (2017), is crucial in simplifying IC_2.

Lemma 3. *For each $t \in T$, the following three statements are equivalent:*

1. *$U_t \in \mathcal{U}$;*

2. *there exists $\pi(\cdot, t, \cdot)$ satisfying IC_2 that gives cost-c agent expected payoff $U_t(c)$;*

3. *there exists an experiment σ that gives cost-c agent expected payoff $U_t(c)$.*

Hence, for any π satisfying IC_2 and $t \in T$, we can let $U_t \in \mathcal{U}$ denote the indirect utility function for type-t agent. Conversely, given $U_L, U_H \in \mathcal{U}$, there exists a persuasion mechanism π satisfying IC_2.

Step 2: Establish the equivalence in principal's objective functions.

Given a persuasion mechanism satisfying IC_2 and its induced U_L and U_H functions, one can follow the argument of Kolotilin et al. (2017) and equivalently write the principal's payoff extracted from type-t agent as follows:

$$\int_{c \in C} \int_{\omega \in \Omega} \pi(\omega, t, c)\phi(\omega) g_t(c) \, \mathrm{d}\omega \, \mathrm{d}c = g_t(0)\mathbb{E}[\omega] + \int_{c \in C} U_t(c) g_t'(c) \, \mathrm{d}c.$$

Hence, the principal's ex ante payoff can be equivalently written as

$$g(0)\mathbb{E}[\omega] + f(L) \int_{c \in C} U_L(c) g_L'(c) \, \mathrm{d}c + f(H) \int_{c \in C} U_H(c) g_H'(c) \, \mathrm{d}c,$$

where the first-term is a constant number.

Step 3: Establish the equivalence between IC_1 and the two inequality constraints in Problem (I).

Recall that IC_1 requires that the agent has no incentive to deviate starting from stage one. Notice that in expression (1), *i.e.*, the IC_2 condition, t only affects a cost-c agent's expected payoff through the π function, and thus, t can also be interpreted as a misreported type. Thus, expression (1) implies that an agent, even if misreported in stage one, will truthfully report and follow recommendation in stage two; that is,

$$U_\pi(\hat{t}, c, 0, 1 | t, c) \geqslant U_\pi(\hat{t}, \hat{c}, a_0, a_1 | t, c)$$

for all $t, c, \hat{t}, \hat{c}, a_0, a_1$. As a result, a persuasion mechanism satisfying IC_2 satisfies IC_1, if and only if the induced U_L and U_H functions satisfy

$$\int_{c \in C} U_t(c) \, \mathrm{d}G_t(c) \geqslant \int_{c \in C} U_{\hat{t}}(c) \, \mathrm{d}G_t(c)$$

for all $t, \hat{t} \in T$.

A.2 Proof of Lemma 2

Given the upper censorship experiment $\sigma_{(0,s)}$, the induced expected utility for an agent with cost c is given as follows:

$$U_{(0,s)}(c) = \begin{cases} (\mathbb{E}[\omega|\omega > c] - c)(1 - \Phi(c)), & c \in [0, s]; \\ (\mathbb{E}[\omega|\omega > s] - c)(1 - \Phi(s)), & c \in \big(s, \mathbb{E}[\omega|\omega > s]\big]; \\ 0, & c \in \big(\mathbb{E}[\omega|\omega > s], 1\big]. \end{cases}$$

The derivative of $\displaystyle\int_{c\in C} U_{(0,s)}(c)g'(c)\,\mathrm{d}c$ with respect to s is

$$\phi(s)\int_{s}^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c.$$

We now identify one optimal s to maximize $\displaystyle\int_{c\in C} U_{(0,s)}(c)g'(c)\,\mathrm{d}c$ by discussing three cases.

Case 1: $\mathbb{E}[\omega|\omega > s] < \hat{s}$. Because $\phi(s)\displaystyle\int_{s}^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c \geqslant 0$, it is weakly profitable to keep increasing s.

Case 2: $s > \hat{s}$. Because $\phi(s)\displaystyle\int_{s}^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c \leqslant 0$, it is weakly profitable to keep decreasing s.

Case 3: $s \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s]$. We now show that $\phi(s)\displaystyle\int_{s}^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c$ satisfies DSCP in s when s increases within this case.

To establish DSCP, suppose

$$\int_{s}^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c \leqslant 0.$$

Now fix any $s' \in C$ such that $s' > s$ and $s' \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s']$.

$$\begin{aligned}
&\int_{s'}^{\mathbb{E}[\omega|\omega>s']}(c-s')g'(c)\,\mathrm{d}c \\
=&\int_{s'}^{\hat{s}}\underbrace{\frac{c-s'}{c-s}}_{\text{increasing in } c}\underbrace{(c-s)g'(c)}_{\geqslant 0}\,\mathrm{d}c + \int_{\hat{s}}^{\mathbb{E}[\omega|\omega>s']}\underbrace{\frac{c-s'}{c-s}}_{\text{increasing in } c}\underbrace{(c-s)g'(c)}_{\leqslant 0}\,\mathrm{d}c \\
\leqslant&\int_{s'}^{\hat{s}}\underbrace{\frac{\hat{s}-s'}{\hat{s}-s}(c-s)g'(c)}_{\geqslant 0}\,\mathrm{d}c + \int_{\hat{s}}^{\mathbb{E}[\omega|\omega>s']}\underbrace{\frac{\hat{s}-s'}{\hat{s}-s}(c-s)g'(c)}_{\leqslant 0}\,\mathrm{d}c
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_s^{\hat{s}} \frac{\hat{s}-s'}{\hat{s}-s}(c-s)g'(c)\,\mathrm{d}c + \int_{\hat{s}}^{\mathbb{E}[\omega|\omega>s]} \frac{\hat{s}-s'}{\hat{s}-s}(c-s)g'(c)\,\mathrm{d}c \\
&=\frac{\hat{s}-s'}{\hat{s}-s}\int_s^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c \leqslant 0.
\end{aligned}
$$

To this end, we have established the DSCP of $\int_s^{\mathbb{E}[\omega|\omega>s]}(c-s)g'(c)\,\mathrm{d}c$ in s when $s \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s]$. Hence, the smallest s^* in $[0, \hat{s}]$ such that $s^* \leqslant \hat{s} \leqslant \mathbb{E}[\omega|\omega > s^*]$ and

$$
\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g'(c)\,\mathrm{d}c \leqslant 0
$$

is optimal.

A.3 A technical result

Lemma 4. *For any function $w : C \to \mathbb{R}$ satisfying the DSCP,*

1. *whenever $\int_0^1 w(c)g_L(c)\,\mathrm{d}c \leqslant 0$ (resp., < 0), it must be true that $\int_0^1 w(c)g_H(x)\,\mathrm{d}c \leqslant 0$ (resp., < 0);*
2. *whenever $\int_0^1 w(c)g_H(c)\,\mathrm{d}c \geqslant 0$ (resp., > 0), it must be true that $\int_0^1 w(c)g_L(x)\,\mathrm{d}c \geqslant 0$ (resp., > 0).*

Proof. Fix any $\hat{x} \in C$ such that $w(c) \geqslant 0$ for all $c \leqslant \hat{x}$ and $w(c) \leqslant 0$ for all $c \geqslant \hat{x}$. Choose $\alpha > 0$ such that $\alpha\frac{g_H(\hat{x})}{g_L(\hat{x})} = 1$. Then for all $x < \hat{x}$, $\alpha\frac{g_H(x)}{g_L(x)} < 1$; for all $x > \hat{x}$, $\alpha\frac{g_H(x)}{g_L(x)} > 1$. Hence,

$$
\begin{aligned}
\int_0^1 w(c)g_H(c)\,\mathrm{d}c &= \frac{1}{\alpha}\int_0^1 w(c)g_L(c)\alpha\frac{g_H(c)}{g_L(c)}\,\mathrm{d}c \\
&= \frac{1}{\alpha}\int_0^{\hat{x}} \underbrace{w(c)g_L(c)}_{\geqslant 0}\underbrace{\alpha\frac{g_H(c)}{g_L(c)}}_{\leqslant 1}\,\mathrm{d}c + \frac{1}{\alpha}\int_{\hat{x}}^1 \underbrace{w(c)g_L(c)}_{\leqslant 0}\underbrace{\alpha\frac{g_H(c)}{g_L(c)}}_{\geqslant 1}\,\mathrm{d}c \\
&\leqslant \frac{1}{\alpha}\int_0^{\hat{x}} w(c)g_L(c)\,\mathrm{d}c + \frac{1}{\alpha}\int_{\hat{x}}^1 w(c)g_L(c)\,\mathrm{d}c \\
&= \frac{1}{\alpha}\int_0^1 w(c)g_L(c)\,\mathrm{d}c.
\end{aligned}
$$

The fact that $\int_0^1 w(c)g_H(c)\,\mathrm{d}c \leqslant \frac{1}{\alpha}\int_0^1 w(c)g_L(c)\,\mathrm{d}c$ directly implies that the lemma holds. □

A.4 Comparative statics for the non-discriminatory solution

In the hypothetical situation that the agent's type $t \in T$ is publically observable, the principal's problem is

$$\max_{U_t \in \mathcal{U}} \int_{c \in C} U_t(c) g_t'(c)\, \mathrm{d}c. \tag{7}$$

By replacing the PDF g with g_t in (2), we can define a crossing point of g_t' as $\hat{s}_t$ (3). Similarly, by replacing the PDF g with g_t in (3), we can define an optimal cutoff of problem (7) as s_t^* (namely, $\sigma_{(0,s_t^*)}$ is an experiment that implements the solution of (7)).

By Assumptions 1 and 2, it is immediate that $\hat{s}_L < \hat{s} < \hat{s}_H$. We now describe the connection between the cutoff values s_L^* and s_H^* as well as s^*.

Lemma 5. *It must be the case that $s_L^* \leqslant s^* \leqslant s_H^*$.*

Proof. In this proof, we utilize the assumption that $\dfrac{g_H}{g_L}$ is strictly increasing to verify that $s_L^* \leqslant s_H^*$. The fact that $s_L^* \leqslant s^*$ and $s^* \leqslant s_H^*$ can be established in a similar way due to the strict monotonicity of $\dfrac{g_H}{g}$ and $\dfrac{g}{g_L}$.

Suppose by way of contradiction that $s_L^* > s_H^*$. According to the definitions of s_H^* and s_L^*, the supposition that $s_L^* > s_H^*$, and the observation that $\hat{s}_L < \hat{s}_H$, we have the following ranking:

$$s_H^* < s_L^* \leqslant \hat{s}_L < \hat{s}_H \leqslant \mathbb{E}[\omega|\omega > s_H^*] < \mathbb{E}[\omega|\omega > s_L^*].$$

Define $w(c) = U_{(0,s_L^*)}(c) - U_{(0,s_H^*)}(c)$, which is a non-negative and single-peak function satisfying $w(0) = w(1) = 0$. Hence, w' satisfies the DSCP. By the definition of s_L^*, *i.e.*, s_L^* is the smallest number in $[0, \hat{s}_L]$ such that $s_L^* \leqslant \hat{s}_L \leqslant \mathbb{E}[\omega|\omega > s_L^*]$ and $\int_{c\in C} U_{(0,s_L^*)}(c) g'(c)\, \mathrm{d}c$ is maximized, and the above ranking, one must have $\int_{c\in C} w(c) g_L'(c)\, \mathrm{d}c > 0$. Then by integration by parts, we have $\int_{c\in C} w'(c) g_L(c)\, \mathrm{d}c = -\int_{c\in C} w(c) g_L'(c)\, \mathrm{d}c < 0$. Now by Lemma 4, $\int_{c\in C} w(c) g_H'(c)\, \mathrm{d}c = -\int_{c\in C} w'(c) g_H(c)\, \mathrm{d}c > 0$. This means that

$$\int_{c\in C} [U_{(0,s_L^*)} - U_{(0,s_H^*)}](c) g_H'(c)\, \mathrm{d}c > 0.$$

A contradiction with the optimality of s_H^*.

To this end, we have established that $s_L^* \leqslant s_H^*$. □

A.5 Proof of Proposition 1

Part (a). Proof for the if direction.

Case I: $s^* > 0$**.** Recall that in this case, (4) holds.

Step 1. We first construct a candidate Lagrangian multiplier $\lambda \geqslant 0$.

Recall that notations s^* and $\hat{s}$ are defined in Section 4.2. Notations s_L^*, s_H^*, $\hat{s}_L$, and $\hat{s}_H$ are defined in Section A.4.

Define

$$\lambda \equiv -f(L)\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c}. \tag{8}$$

We first show that $\lambda \geqslant 0$.

By Lemma 5, $s_L^* \leqslant s^*$, implying that $\mathbb{E}[\omega|\omega > s_L^*] \leqslant \mathbb{E}[\omega|\omega > s^*]$.

If $s^* \geqslant \hat{s}_L$, then

$$\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c \leqslant 0,$$

because g_L' is non-positive for $c \geqslant \hat{s}_L$.

If $s^* < \hat{s}_L$, then it must be the case that

$$s_L^* \leqslant s^* < \hat{s}_L < \mathbb{E}[\omega|\omega > s_L^*] \leqslant \mathbb{E}[\omega|\omega > s^*].$$

Following the same argument as in Case 3 in the proof of Lemma 2, we can show that

$$\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c \leqslant 0.$$

Hence, the right hand side of (8) is nonnegative, implying that $\lambda \geqslant 0$. This completes the analysis of Step 1.

Step 2. Given the above λ, we show that among all candidates in $\mathcal{U}$, $U_{(0,s^*)}$ maximizes $\mathcal{L}_1$.

Since $\left[f(L)\frac{g_L'(c)}{g_L(c)} + \lambda\right] g_L(c)$ satisfies DSCP, the optimal U_L can be implemented by an upper censorship experiment.

By (8), we have

$$\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\left[f(L)\frac{g_L'(c)}{g_L(c)} + \lambda\right] g_L(c)\,\mathrm{d}c = 0.$$

Because of the monotonicity of $\frac{g_L'}{g_L}$ and the definition of λ, $\left[f(L)\frac{g_L'(c)}{g_L(c)}+\lambda\right]g_L(c)$ must be nonnegative when evaluated at $c \leqslant s^*$ and nonpositive when evaluated at $c \geqslant \mathbb{E}[\omega|\omega > s^*]$. We can modify the argument in Lemma 2 and show that for all $s < s^*$ (resp. $s > s^*$),

$$\int_s^{\mathbb{E}[\omega|\omega>s]}(c-s)\left[f(L)\frac{g_L'(c)}{g_L(c)}+\lambda\right]g_L(c)\,\mathrm{d}c \geqslant 0 \text{ (resp. } \leqslant 0).$$

Hence, the optimal U_L to maximize $\mathcal{L}_1$ can be implemented by $\sigma_{(0,s^*)}$.

This completes the analysis of Step 2.

Step 3. Show that given λ, among all candidates in $\mathcal{U}$, $U_{(0,s^*)}$ maximizes $\mathcal{L}_2$. To do this, we first identify an upper bound of $\mathcal{L}_2$ when $U_H \in \mathcal{U}$. Then we show that $U_H = U_{(0,s^*)}$ attains this upper bound.

Given this λ,

$$\begin{aligned}
\mathcal{L}_2 =& \int_{c\in C} U_H(c)\left[f(H)\frac{g_H'(c)}{g_L(c)}-\lambda\right]g_L(c)\,\mathrm{d}c \\
=& f(H)\int_{c\in C} U_H(c)\left[\underbrace{g_H'(c)+\frac{f(L)}{f(H)}\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L'(\tilde{c})\,\mathrm{d}\tilde{c}}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L(\tilde{c})\,\mathrm{d}\tilde{c}}\cdot g_L(c)}_{\equiv\Psi(c)}\right]\mathrm{d}c \\
=& f(H)\int_0^{s^*} U_H(c)\Psi(c)\,\mathrm{d}c + f(H)\int_{s^*}^1 U_H(c)\Psi(c)\,\mathrm{d}c \\
=& f(H)\int_0^{s^*} U_H(c)\Psi(c)\,\mathrm{d}c - f(H)\left[\int_{s^*}^1 U_H(c)\,\mathrm{d}\int_c^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c}\right] \\
=& f(H)\int_0^{s^*} U_H(c)\Psi(c)\,\mathrm{d}c - f(H)\Bigg[\underbrace{U_H(1)}_{=0}\int_1^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c} \\
&- U_H(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c} - \int_{s^*}^1\int_c^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c}\cdot U_H'(c)\,\mathrm{d}c\Bigg].
\end{aligned} \tag{9}$$

Notice that

$$\begin{aligned}
&\int_{s^*}^1\int_c^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c}\cdot U_H'(c)\,\mathrm{d}c \\
=& -\int_{s^*}^1 U_H'(c)\,\mathrm{d}\int_c^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-c)\Psi(\hat{c})\,\mathrm{d}\hat{c} \\
=& -\underbrace{U_H'(1)}_{=0}\int_1^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-1)\Psi(\hat{c})\,\mathrm{d}\hat{c} + U_H'(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-s^*)\Psi(\hat{c})\,\mathrm{d}\hat{c}
\end{aligned}$$

$$+\int_{s^*}^1 \underbrace{\int_c^{\mathbb{E}[\omega|\omega>s^*]} (\hat{c}-c)\Psi(\hat{c})\,\mathrm{d}\hat{c}}_{\leqslant 0 \text{ as we show below}} \cdot \underbrace{U_H''(c)}_{\geqslant 0}\,\mathrm{d}c$$
$$\leqslant U_H'(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (\hat{c}-s^*)\Psi(\hat{c})\,\mathrm{d}\hat{c}.$$

The inequality follows from the observation that

$$\int_c^{\mathbb{E}[\omega|\omega>s^*]} (\hat{c}-c)\Psi(\hat{c})\,\mathrm{d}\hat{c}$$
$$=\int_c^{\mathbb{E}[\omega|\omega>s^*]} (\hat{c}-c)\left[g_H'(\hat{c})+\frac{f(L)}{f(H)}\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L'(\tilde{c})\,\mathrm{d}\tilde{c}}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L(\tilde{c})\,\mathrm{d}\tilde{c}}\cdot g_L(\hat{c})\right]\mathrm{d}\hat{c}$$
$$=\left[\frac{\int_c^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-c)g_H'(\hat{c})\,\mathrm{d}\hat{c}}{\int_c^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-c)g_L(\hat{c})\,\mathrm{d}\hat{c}}+\frac{f(L)}{f(H)}\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L'(\tilde{c})\,\mathrm{d}\tilde{c}}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L(\tilde{c})\,\mathrm{d}\tilde{c}}\right]\int_c^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-c)g_L(\hat{c})\,\mathrm{d}\hat{c},$$

which is equal to 0 for $c=s^*$ by (4) and is nonpositive for $c>s^*$ by (5).

Thus,

(9)
$$\leqslant f(H)\left[\int_0^{s^*} U_H(c)\underbrace{\Psi(c)}_{\geqslant 0 \text{ for } c<s^* \text{ by (5)}}\mathrm{d}c+U_H(s^*)\underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}\Psi(c)\,\mathrm{d}c}_{\geqslant 0 \text{ as shown below}}+U_H'(s^*)\underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\Psi(c)\,\mathrm{d}c}_{=0}\right]$$
$$\leqslant f(H)\left[\int_0^{s^*}\overline{U}(c)\Psi(c)\,\mathrm{d}c+\overline{U}(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}\Psi(c)\,\mathrm{d}c\right].$$

The second inequality follows from the observation above that $\int_c^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-c)\Psi(\hat{c})\,\mathrm{d}\hat{c}$ takes the value 0 and is nonpositive when $c>s^*$, and thus, $-\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c}$, its derivative at $c=s^*$, is no greater than 0.

To this end, we have established an upper bound of $\mathcal{L}_2$ when $U_H\in\mathcal{U}$.

Now we want to show that $U_H=U_{(0,s^*)}$ attains the above upper bound of $\mathcal{L}_2$. When $U_H=U_{(0,s^*)}$, $U_H'(c)$ is constantly equal to $\overline{U}_H'(s^*)$ on $[s^*,\mathbb{E}[\omega|\omega>s^*]]$, and is equal to 0 on $(\mathbb{E}[\omega|\omega>s^*],1]$. As a result,

$$\int_{s^*}^1\int_c^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}c\cdot U_H'(c)\,\mathrm{d}c=\overline{U}_H'(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}\int_c^{\mathbb{E}[\omega|\omega>s^*]}\Psi(\hat{c})\,\mathrm{d}\hat{c}\,\mathrm{d}c$$
$$=\overline{U}_H'(s^*)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\hat{c}-s^*)\Psi(\hat{c})\,\mathrm{d}\hat{c}=0,$$

where the second equality is derived by changing the order of integration. Hence,

$$\mathcal{L}_2 = f(H)\left[\int_0^{s*} \overline{U}(c)\Psi(c)\,\mathrm{d}c + \overline{U}(s^*)\int_{s*}^{\mathbb{E}[\omega|\omega>s*]} \Psi(c)\,\mathrm{d}c\right].$$

Therefore, $U_H = U_{(0,s^*)}$ attains the above upper bound of $\mathcal{L}_2$. It completes the analysis of Step 3.

Case II: $s^* = 0$.

We follow the argument of Case I till establishing an upper bound of $\mathcal{L}_2$ in Step 3:

$$\begin{aligned}(9) \leqslant & f(H)\underbrace{U_H(0)}_{=\mathbb{E}[\omega]}\int_0^{\mathbb{E}[\omega]} \Psi(c)\,\mathrm{d}c + f(H)\underbrace{U'_H(0)}_{=-1}\int_0^{\mathbb{E}[\omega]} (c-0)\Psi(c)\,\mathrm{d}c \\ = & f(H)\int_0^{\mathbb{E}[\omega]} (\mathbb{E}[\omega]-c)\Psi(c)\,\mathrm{d}c.\end{aligned}$$

To show that $U_H = U_{(0,0)}$ attains the above upper bound of $\mathcal{L}_2$, notice that

$$\begin{aligned}\mathcal{L}_2 = & \int_{c\in C} U_H(c)\left[f(H)\frac{g'_H(c)}{g_L(c)} - \lambda\right] g_L(c)\,\mathrm{d}c \\ = & f(H)\int_0^1 U_H(c)\Psi(c)\,\mathrm{d}c \\ = & f(H)\int_0^{\mathbb{E}[\omega]} (\mathbb{E}[\omega]-c)\Psi(c)\,\mathrm{d}c.\end{aligned}$$

Notice that the second and third equalities utilize the definitions of $\Psi(c)$ and $U_{(0,0)}$.

To this end, we have established the proof for Case II.

Part (a). Proof for the only if direction.

We show that if condition (5) fails, then we can construct a menu of experiments that performs better than the single experiment $\sigma_{(0,s^*)}$.

Case I: Suppose that there is some $c_1 < s^*$ such that the LHS inequality in (5) is violated, or equivalently, $\Psi(c_1) < 0$, where

$$\Psi(c) \equiv g'_H(c) + \frac{f(L)}{f(H)} \frac{\int_{s*}^{\mathbb{E}[\omega|\omega>s*]}(c-s^*)g'_L(c)\,\mathrm{d}c}{\int_{s*}^{\mathbb{E}[\omega|\omega>s*]}(c-s^*)g_L(c)\,\mathrm{d}c} \cdot g_L(c). \tag{10}$$

It is immediate that $s^* > 0$.

As we descibe in the text, let a new experiment designed for the L type be $\sigma_{(0,s*-\epsilon)}$ and construct a new experiment for H type as follows:

- fully revealing on $[0, c_1] \cup [c_1 + \delta_1(\epsilon), s^*]$;
- pooling on $(c_1, c_1 + \delta_1(\epsilon))$;
- pooling on $(s^*, 1]$.

The choice of $\delta_1(\epsilon)$ is such that

$$\int_{c\in C} U_{(0,s^*-\epsilon)}(c) g_L(c)\,\mathrm{d}c = \int_{c\in C} U_{\delta_1(\epsilon)}(c) g_L(c)\,\mathrm{d}c \tag{11}$$

and $\epsilon > 0$ should be sufficiently small such that $s^* - \epsilon > 0$ and $c_1 + \delta_1(\epsilon) < s^*$, where $U_{\delta_1(\epsilon)}(c)$ refers to the expected utility cost-c agent receives from experiment designed for the H type.

The expression of $U_{\delta_1(\epsilon)}(c)$ is given as follows:

$$U_{\delta_1(\epsilon)}(c) = \begin{cases} (\mathbb{E}[\omega|\omega \geqslant c] - c)(1-\Phi(c)), & c \in [0, c_1); \\ (\mathbb{E}[\omega|\omega \geqslant c_1] - c)(1-\Phi(c_1)), & c \in \big[c_1, \mathbb{E}[\omega|c_1 \leqslant \omega < c_1 + \delta_1(\epsilon)]\big); \\ (\mathbb{E}[\omega|\omega \geqslant c_1 + \delta_1(\epsilon)] - c)(1-\Phi(c_1+\delta_1(\epsilon))), & c \in \big[\mathbb{E}[\omega|c_1 \leqslant \omega < c_1 + \delta_1(\epsilon)], c_1 + \delta_1(\epsilon)\big); \\ (\mathbb{E}[\omega|\omega \geqslant c] - c)(1-\Phi(c)), & c \in [c_1 + \delta_1(\epsilon), s^*); \\ (\mathbb{E}[\omega|\omega \geqslant s^*] - c)(1-\Phi(s^*)), & c \in \big[s^*, \mathbb{E}[\omega|\omega \geqslant s^*]\big); \\ 0, & c \in \big[\mathbb{E}[\omega|\omega \geqslant s^*], 1\big], \end{cases}$$

The principal's payoff under the menu of experiments is equal to

$$g(0)\mathbb{E}[\omega] + f(L) \int_{c\in C} U_{(0,s^*-\epsilon)}(c) g'_L(c)\,\mathrm{d}c + f(H) \int_{c\in C} U_{\delta_1(\epsilon)}(c) g'_H(c)\,\mathrm{d}c. \tag{12}$$

The partial derivative of (12) with respect to ϵ is

$$-f(L)\phi(s^*-\epsilon) \int_{s^*-\epsilon}^{\mathbb{E}[\omega|\omega>s^*-\epsilon]} (c - s^* + \epsilon) g'_L(c)\,\mathrm{d}c.$$

The partial derivative of (12) with respect to $\delta_1(\epsilon)$, is given by

$$-f(H)\phi(c_1+\delta_1(\epsilon)) \int_{\mathbb{E}[\omega|\omega\in(c_1,c_1+\delta_1(\epsilon))]}^{c_1+\delta_1(\epsilon)} (c_1 + \delta_1(\epsilon) - c) g'_H(c)\,\mathrm{d}c.$$

By applying the implicit function theorem to (11) we have

$$\delta'_1(\epsilon) = \frac{\phi(s^*-\epsilon)\int_{s^*-\epsilon}^{\mathbb{E}[\omega|\omega>s^*-\epsilon]}(c - s^* + \epsilon) g_L(c)\,\mathrm{d}c}{\phi(c_1+\delta_1(\epsilon))\int_{\mathbb{E}[\omega|\omega\in(c_1,c_1+\delta_1(\epsilon))]}^{c_1+\delta_1(\epsilon)}(c_1 + \delta_1(\epsilon) - c) g_L(c)\,\mathrm{d}c}.$$

Putting the above three parts together, the derivative of (14) with respect to ϵ, evaluated at $\epsilon = 0$, can be rewritten as $\phi(s^*)$ times

$$\begin{aligned}
&- f(L)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c \\
&- f(H)\frac{\int_{\mathbb{E}[\omega|\omega\in(c_1,c_1+\delta_1(\epsilon))]}^{c_1+\delta_1(\epsilon)}(c_1+\delta_1(\epsilon)-c)g_H'(c)\,\mathrm{d}c|_{\delta_1(\epsilon)\to 0}}{\int_{\mathbb{E}[\omega|\omega\in(c_1,c_1+\delta_1(\epsilon))]}^{c_1+\delta_1(\epsilon)}(c_1+\delta_1(\epsilon)-c)g_L(c)\,\mathrm{d}c|_{\delta_1(\epsilon)\to 0}}\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c \\
=& - f(L)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c - f(H)\frac{g_H'(c_1)}{g_L(c_1)}\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c > 0,
\end{aligned}$$

where the equality follows from the L'Hospital Rule and the inequality follows from (10).

Case II: Suppose that there is some $c_2 > s^*$ such that the RHS inequality in (5) is violated. It is clear that $s^* < 1$. By the definition of Ψ in (10), it is equivalent to assume that

$$\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)\Psi(c)\,\mathrm{d}c > 0.$$

Without loss, suppose that $c_2 < 1$.

As described in the text, let the new experiment designed for type L be $\sigma_{(0,s^*+\epsilon)}$ and the one for H type be:

- fully revealing on $[0, s^*] \cup [c_2, c_2 + \delta_2(\epsilon)]$;
- pooling on $(s^*, c_2) \cup (c_2 + \delta_2(\epsilon), 1]$.

Again, the choice of $\delta_2(\epsilon)$ is such that

$$\int_{c\in C} U_{(0,s^*+\epsilon)}(c)g_L(c)\,\mathrm{d}c = \int_{c\in C} U_{\delta_2(\epsilon)}(c)g_L(c)\,\mathrm{d}c \tag{13}$$

and ϵ should be sufficiently small such that $s^* + \epsilon < 1$ and $c_2 + \delta_2(\epsilon) < 1$, where $U_{\delta_2(\epsilon)}(c)$ refers to the expected utility cost-c agent receives from experiment designed for the H type.

Expression $U_{\delta_2(\epsilon)}(c)$ can take one of the following three forms specified below.

1. If $X \equiv \mathbb{E}\big[\omega | \omega \in [s^*, c_2) \cup (c_2 + \delta_2(\epsilon), 1]\big] \leqslant c_2$, then

$$U_{\delta(\epsilon)}(c) = \begin{cases} \int_c^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [0, s^*); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [s^*, X); \\ \int_{c_2}^{c_2+\delta_2(\epsilon)} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [X, c_2); \\ \int_c^{c_2+\delta_2(\epsilon)} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [c_2, c_2 + \delta_2(\epsilon)]; \\ 0, & c \in [c_2 + \delta_2(\epsilon), 1]. \end{cases}$$

2. If $c_2 < X \leqslant c_2 + \delta_2(\epsilon)$, then

$$U_{\delta_2(\epsilon)}(c) = \begin{cases} \int_c^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [0, s^*); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [s^*, c_2); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega - \int_{c_2}^{c} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [c_2, X); \\ \int_c^{c_2+\delta_2(\epsilon)} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [X, c_2 + \delta_2(\epsilon)); \\ 0, & c \in [c_2 + \delta_2(\epsilon), 1]. \end{cases}$$

3. If $X > c_2 + \delta_2(\epsilon)$, then

$$U_{\delta_2(\epsilon)}(c) = \begin{cases} \int_c^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [0, s^*); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [s^*, c_2); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega - \int_{c_2}^{c} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [c_2, c_2 + \delta_2(\epsilon)); \\ \int_{s^*}^1 (\omega - c)\phi(\omega)\,\mathrm{d}\omega - \int_{c_2}^{c_2+\delta_2(\epsilon)} (\omega - c)\phi(\omega)\,\mathrm{d}\omega, & c \in [c_2 + \delta_2(\epsilon), X); \\ 0, & c \in [X, 1]. \end{cases}$$

The partial derivative of the following expression

$$g(0)\mathbb{E}[\omega] + f(L)\int_{c\in C} U_{(0,s^*+\epsilon)}(c) g_L'(c)\,\mathrm{d}c + f(H)\int_{c\in C} U_{\delta_2(\epsilon)}(c) g_H'(c)\,\mathrm{d}c, \tag{14}$$

with respect to ϵ is

$$f(L)\phi(s^* + \epsilon)\int_{s^*+\epsilon}^{\mathbb{E}[\omega|\omega > s^*+\epsilon]} (c - s^* - \epsilon) g_L'(c)\,\mathrm{d}c.$$

The partial derivative of (14) with respect to $\delta_2(\epsilon)$ is given by

$$f(H)\phi(c_2+\delta_2(\epsilon))\int_X^{c_2+\delta_2(\epsilon)}(c_2+\delta_2(\epsilon)-c)g'_H(c)\,\mathrm{d}c.$$

By applying the implicit function theorem to (13), we have

$$\delta'_2(\epsilon)=\frac{\phi(s^*+\epsilon)\int_{s^*+\epsilon}^{\mathbb{E}[\omega|\omega>s^*+\epsilon]}(c-s^*-\epsilon)g_L(c)\,\mathrm{d}c}{\phi(c_2+\delta_2(\epsilon))\int_X^{c_2+\delta_2(\epsilon)}(c_2+\delta_2(\epsilon)-c)g_L(c)\,\mathrm{d}c}.$$

Putting the above three parts together, the derivative of (14) with respect to ϵ, evaluated at $\epsilon=0$, can be rewritten as $\phi(s^*)$ times

$$f(L)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g'_L(c)\,\mathrm{d}c+f(H)\frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g'_H(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g_L(c)\,\mathrm{d}c}\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c>0.$$

In both cases, we have shown that introducing information discrimination is strictly more profitable than adopting the optimal non-discriminatory persuasion.

Part (b). Omitted proof.

Step 1. Fix any (U_L,U_H) in $\mathcal{U}$ that satisfies the upward IC_1 constraint in Problem (II). We now construct $\hat{U}_L$ such that

1. there exists $c_L\in[0,1]$, for which $\sigma_{(0,c_L)}$ implements $\hat{U}_L$, namely $\hat{U}_L=U_{(0,c_L)}$;
2. the upward IC_1 constraint is satisfied at $(\hat{U}_L,U_H)$;
3. the principal gets a weakly higher payoff from $(\hat{U}_L,U_H)$ than from (U_L,U_H).

To construct $\hat{U}_L$, we solve **Problem (III)** below.

$$\max_{\hat{U}_L\in\mathcal{U}}\int_{c\in C}\hat{U}_L(c)g'_L(c)\,\mathrm{d}c$$
$$\text{s.t.}\int_{c\in C}\hat{U}_L(c)g_L(c)\,\mathrm{d}c\geqslant\int_{c\in C}U_H(c)g_L(c)\,\mathrm{d}c.$$

Let c_L be such that $\int_{c\in C}U_{(0,c_L)}(c)g_L(c)\,\mathrm{d}c=\int_{c\in C}U_H(c)g_L(c)\,\mathrm{d}c.$

If $c_L\leqslant s^*_L$, then we let $\hat{U}_L$ be $U_{(0,s^*_L)}$, which satisfies the upward IC_1 constraint. Because $U_{(0,s^*_L)}$ maximizes the objective function even without any constraint, it must be the solution of Problem (III).

If $c_L > s_L^*$ and thus, $c_L > 0$, then we let $\hat{U}_L$ be $U_{(0,c_L)}$. To see that $U_{(0,c_L)}$ is optimal, define

$$\lambda \equiv -\frac{\int_{c_L}^{\mathbb{E}[\omega|\omega>c_L]}(c-c_L)g_L'(c)\,\mathrm{d}c}{\int_{c_L}^{\mathbb{E}[\omega|\omega>c_L]}(c-c_L)g_L(c)\,\mathrm{d}c},$$

which can be shown to be nonnegative. It is clear that the coefficient of $\hat{U}_L$ in the Lagrangian of Problem (III), $g_L'+\lambda g_L$, exhibits DSCP. As a result, following Proposition 1, the experiment that implements the optimal $\hat{U}_L$ must involve upper censorship. Given the above λ, the verification of the first order condition and the second order condition concludes that $\hat{U}_L = U_{(0,c_L)}$ is indeed a solution of Problem (III).

Step 2. Show that the upward IC_1 constraint must bind in Problem (II).

In the Lagrangian of Problem (II), we assume by way of contradiction that the upward IC_1 constraint is slack. In this case, $\lambda = 0$. Then first and second order conditions in $\mathcal{L}_1$ and $\mathcal{L}_2$ show that $U_L = U_{(0,s_L^*)}$ and $U_H = U_{(0,s_H^*)}$ constitute one optimal solution. By the observation that $s_L^* \leqslant s_H^*$ from Lemma 5, the upward IC_1 constraint is either violated or binding. This is a contradiction with the supposition that the constraint is slack.

Step 3. Show that the solution of Problem (II) solves Problem (I).

It suffices to show that the binding upward IC_1 constraint in Problem (II), *i.e.*,

$$\int_{c\in C} U_{(0,c_L)}(c)g_L(c)\,\mathrm{d}c = \int_{c\in C} U_H(c)g_L(c)\,\mathrm{d}c$$

implies the downward IC_1 constraint in Problem (I), *i.e.*,

$$\int_{c\in C} U_{(0,c_L)}(c)g_H(c)\,\mathrm{d}c \leqslant \int_{c\in C} U_H(c)g_H(c)\,\mathrm{d}c.$$

Towards this end, we shall show that the function $U_{(0,c_L)} - U_H$ satisfies DSCP. We prove by contradiction. Suppose that there exist $s_1 < s_2$ such that that

$$U_{(0,c_L)}(s_1) < U_H(s_1), \qquad U_{(0,c_L)}(s_2) > U_H(s_2).$$

By the definition of $U_{(0,c_L)}$, $c_L < s_1 < s_2 < \mathbb{E}[\omega|\omega > c_L]$ and

$$U_{(0,c_L)}(c_L) \geqslant U_H(c_L). \tag{15}$$

Then we have

$$
\begin{aligned}
&U_H(c_L) + U_H'(s_1)(s_1 - c_L) \\
\geqslant &U_H(s_1) \\
> &U_{(0,c_L)}(s_1) \\
= &U_{(0,c_L)}(c_L) + U_{(0,c_L)}'(s_1)(s_1 - c_L). \qquad (16)
\end{aligned}
$$

The first inequality holds because U_H is convex. The equality holds as $U_{(0,c_L)}$ is linear between c_L and $\mathbb{E}[\omega|\omega > c_L]$. Expressions (15) and (16) imply that $U_H'(s_1) > U_{(0,c_L)}'(s_1)$. Then we have that

$$
\begin{aligned}
&U_H(s_2) \\
\geqslant &U_H(s_1) + U_H'(s_1)(s_2 - s_1) \\
> &U_{(0,c_L)}(s_1) + U_{(0,c_L)}'(s_1)(s_2 - s_1) \\
= &U_{(0,c_L)}(s_2).
\end{aligned}
$$

A contraction. Thus, $U_{(0,c_L)} - U_H$ satisfies DSCP. The downward IC_1 constraint follows from Lemma 4.

A.6 Proof of Corollary 1

Part (a). If $s^* = 0$, then (5) equivalently requires that for all $c_2 > s^*$,

$$
-\frac{f(L)}{f(H)} \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L(c)\,\mathrm{d}c} \geqslant \frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c - c_2)g_H'(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c - c_2)g_L(c)\,\mathrm{d}c}.
$$

But $s^* = 0$ implies that

$$
-\frac{f(L)}{f(H)} \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L(c)\,\mathrm{d}c} \geqslant \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_H'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L(c)\,\mathrm{d}c}.
$$

As a result, it sufficies to show that

$$
\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_H'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c - s^*)g_L(c)\,\mathrm{d}c} \geqslant \frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c - c_2)g_H'(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c - c_2)g_L(c)\,\mathrm{d}c}, \qquad (17)
$$

which is part of (6).

If $s^* > 0$, then (5) is equivalent to (6).

Hence, in both cases, it sufficies to show that (6) holds.

Assume that $\frac{g_H'(c)}{g_L(c)}$ is decreasing in $c \in C$. Obviously, the first inequality in (6) holds, *i.e.*, for all $c_1 < s^*$,

$$\frac{g_H'(c_1)}{g_L(c_1)} \geqslant \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_H'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c}.$$

To show that the second inequality in (6) holds, it is equivalent to show that for

$$\Psi(c) \overset{(4)(10)}{=} g_H'(c) - \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_H'(\tilde{c})\,\mathrm{d}\tilde{c}}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(\tilde{c}-s^*)g_L(\tilde{c})\,\mathrm{d}\tilde{c}} g_L(c), \forall c \in C, \tag{18}$$

one must have

$$\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)\Psi(c)\,\mathrm{d}c \leqslant 0, \forall c_2 \in (s^*, 1].$$

By definition of Ψ,

$$\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\Psi(c)\,\mathrm{d}c = 0.$$

Also, by the assumption that $\frac{g_H'(c)}{g_L(c)}$ is decreasing, $\frac{\Psi(c)}{g_L(c)}$ is decreasing in $c \in C$. This further implies that Ψ must satisfy the DSCP. Let $\hat{c}$ be one crossing point of Ψ identified similarly as in (2). Then it must be the case that $s^* < \hat{c}$.

For $c_2 \in (s^*, \hat{c}]$,

$$\begin{aligned}
&\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)\Psi(c)\,\mathrm{d}c \\
=&\int_{c_2}^{\hat{c}}(c-s^*)\cdot\frac{c-c_2}{c-s^*}\underbrace{\Psi(c)}_{\geqslant 0}\,\mathrm{d}c + \int_{\hat{c}}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\cdot\frac{c-c_2}{c-s^*}\underbrace{\Psi(c)}_{\leqslant 0}\,\mathrm{d}c \\
\leqslant&\int_{c_2}^{\hat{c}}(c-s^*)\cdot\frac{\hat{c}-c_2}{\hat{c}-s^*}\Psi(c)\,\mathrm{d}c + \int_{\hat{c}}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\cdot\frac{\hat{c}-c_2}{\hat{c}-s^*}\Psi(c)\,\mathrm{d}c \\
=&\frac{\hat{c}-c_2}{\hat{c}-s^*}\cdot\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\Psi(c)\,\mathrm{d}c \\
\leqslant&\frac{\hat{c}-c_2}{\hat{c}-s^*}\cdot\underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)\Psi(c)\,\mathrm{d}c}_{=0} \\
=&0.
\end{aligned}$$

For $c_2 \in \big(\hat{c}, \mathbb{E}[\omega|\omega > s^*]\big]$, it is obvious that

$$\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]} \underbrace{(c-c_2)}_{\geqslant 0} \underbrace{\Psi(c)}_{\leqslant 0} \,\mathrm{d}c \leqslant 0.$$

For $c_2 \in \big(\mathbb{E}[\omega|\omega > s^*], 1\big]$, notice that the lower limit of the integration below is higher than the upper limit, and thus

$$\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]} \underbrace{(c-c_2)}_{\leqslant 0} \underbrace{\Psi(c)}_{\leqslant 0} \,\mathrm{d}c \leqslant 0.$$

As a result, the second inequality of (6) also holds.

In sum, (6) holds, which implies that the optimal persuasion mechanism can be implemented by one experiment.

Part (b).

Assume that (4) holds and $\dfrac{g_H'(c)}{g_L(c)}$ is strictly increasing in $c \in C$.

If $s^* > 0$, then obviously, for all $c_1 < s^*$,

$$\frac{g_H'(c_1)}{g_L(c_1)} < \frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_H'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c} \overset{(4)}{=} -\frac{f(L)}{f(H)}\frac{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L'(c)\,\mathrm{d}c}{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_L(c)\,\mathrm{d}c}.$$

Namely, the first inequality in (5) is violated.

Now we assume that $s^* = 0$. We can see that $\dfrac{\Psi(c)}{g_L(c)}$, where Ψ defined in (10) and equivalently in (18) given (4), is strictly increasing in $c \in C$, negative when $c = 0$, and positive when $c \geqslant \mathbb{E}[\omega]$. Hence, there exists a unique point $\hat{c} \in (0, \mathbb{E}[\omega])$ such that $\Psi(\hat{c}) = 0$, $\Psi(c) < 0$ for $c < \hat{c}$, and $\Psi(c) > 0$ for $c > \hat{c}$. it must be the case that $s^* < \hat{c} < \mathbb{E}[\omega|\omega > s^*]$. For $c_2 \in (s^*, \hat{c}]$, we can reverse the argument above and show that

$$\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]} (c-c_2)\Psi(c)\,\mathrm{d}c > 0.$$

The above inequality along with (10) implies that the second inequality in (5) is violated for c_2.

In sum, the optimal persuasion mechanism cannot be implemented by one experiment.

B Online Appendix B: Multiple types

We now consider the case where there are more than two types. Let $T = \{t_1, t_2, ..., t_K\}$ where $K \geqslant 3$. For each $k = 1, 2, ..., K$, we slightly abuse notation by letting $f(k)$ denote the probability that t_k is realized, and $G_k(\cdot)$ and $g_k(\cdot)$ be the CDF and PDF of c conditional on t_k in stage one. We now restate Assumption 1 as follows: For each $k \in \{1, 2, ..., K\}$, PDF $g_k(c)$ and PDF $g(c)$ are log-concave in c. Assumption 2 is restated as follows: The likelihood ratio $\dfrac{g_{k+1}(c)}{g_k(c)}$ is strictly increasing in c for each $k \in \{1, 2, ..., K-1\}$.

As before, Problem (O) is equivalent to the following **Problem (I)**.

$$\max_{(U_k \in \mathcal{U})_{k=1,\dots,K}} \quad \sum_{k=1,\dots,K} f(k) \int_{c\in C} U_k(c) g_k'(c)\, \mathrm{d}c$$
$$\text{s.t.} \quad \int_{c\in C} U_k(c) g_k(c)\, \mathrm{d}c \geqslant \int_{c\in C} U_{k'}(c) g_k(c)\, \mathrm{d}c, \forall k, k' \in \{1, ..., K\}.$$

B.1 Sufficient condition for non-discriminatory disclosure

For each $k \in \{1, ..., K\}$, let s_k^* be defined in a similar way as (3) but under PDF g_k such that $U_{(0,s_k^*)}$ solve the following problem

$$\max_{U\in\mathcal{U}} \int_{c\in C} U(c) g_k'(c). \tag{19}$$

Namely, $U_{(0,s_k^*)}$ maximizes the surplus extracted from type-t_k agent.

We now present a sufficient condition for Problem (I) to have a non-discriminatory solution $(U_k = U_{(0,s^*)})_{k\in\{1,\dots,K\}}$.

Proposition 2. *The optimal persuasion mechanism can be implemented by one experiment if* $\dfrac{g_k'(c)}{g_1(c)}$ *is decreasing for all* $k \in \{1, ..., K\}$ *such that* $s_k^* \geqslant s^*$.

Proof of Proposition 2. Fix a pair of nonempty sets κ^-, $\kappa^+ \subseteq \{1, ..., K\}$ such that $\kappa^- \cup \kappa^+ = \{1, ..., K\}$. Consider a relaxed variant of Problem (I) where only constraints $IC(t_k, t_{k'})$, *i.e.*, t_k should not have the incentive to misreport $t_{k'}$, for all $k \in \kappa^-$ and $k' \in \kappa^+$ are imposed. If we show that $(U_k = U_{(0,s^*)})_{k\in\{1,\dots,K\}}$ solves the relaxed problem, then it is obvious that it also solves Problem (I) because the omitted constraints are trivially satisfied. The Lagrangian of the relaxed problem is given as follows.

$$\mathcal{L} = \sum_{k=1,\dots K} f(k) \int_{c\in C} U_k(c) g_k'(c)\, \mathrm{d}c + \sum_{k\in\kappa^-, k'\in\kappa^+} \lambda_{k,k'} \int_{c\in C} \left[U_k(c) - U_{k'}(c)\right] g_k(c)\, \mathrm{d}c = \sum_{k=1,\dots,K} \mathcal{L}_k,$$

where

$$\mathcal{L}_k \equiv \int_{c\in C} U_k(c)\left[f(k)\frac{g_k'(c)}{g_k(c)} + \sum_{k'\in\kappa^+}\lambda_{k,k'}\right] g_k(c)\,\mathrm{d}c, \forall k \in \kappa^-,$$

$$\mathcal{L}_{k'} \equiv \int_{c\in C} U_{k'}(c)\left[f(k')\frac{g_{k'}'(c)}{g_1(c)} - \sum_{k\in\kappa^-}\lambda_{k,k'}\frac{g_k(c)}{g_1(c)}\right] g_1(c)\,\mathrm{d}c, \forall k' \in \kappa^+.$$

We now proceed to construct the sets κ^- and κ^+ and solve the relaxed problem by discussing three cases.

Case 1. Suppose $s^* \in (0,1)$.

Recall that in this case,

$$\sum_{k\in\{1,\ldots,K\}} f(k)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_k'(c)\,\mathrm{d}c = 0.$$

Let $\kappa^- \equiv \{1\}\cup\{k\in\{2,...,K-1\} : s_k^* < s^*\}$ and $\kappa^+ \equiv \{k\in\{2,...,K-1\} : s_k^* \geqslant s^*\}\cup\{K\}$. Both sets are nonempty. Let K^- and K^+ denote the cardinality of κ^- and κ^+, respectively.

Given any nonnegative profile of Lagrangian multiplier $(\lambda_{k,k'})_{k\in\kappa^-,k'\in\kappa^+}$, for each $k\in\kappa^-$,

$$f(k)\frac{g_k'(c)}{g_k(c)} + \sum_{k'\in\kappa^+}\lambda_{k,k'}$$

is decreasing by Assumption 1 (stated for $K \geqslant 3$), and for each $k'\in\kappa^+$,

$$f(k')\frac{g_{k'}'(c)}{g_1(c)} - \sum_{k\in\kappa^-}\lambda_{k,k'}\frac{g_k(c)}{g_1(c)}$$

is decreasing by Assumption 2 and the sufficient condition stated in Proposition 2.

As a result, these expressions above satisfy the DSCP.

We now take four steps to solve this relaxed problem.

Step 1. We introduce a few notations.

For each positive interger $k \in \kappa^-$, define a $K^- \times K^+$ matrix:

$$\mathbf{B}^k_{K^- \times K^+} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k(c)\, \mathrm{d}c & \dots & \int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k(c)\, \mathrm{d}c \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \leftarrow k\text{-th row}$$

and a $K^+ \times K^+$ matrix:

$$\mathbf{C}^k_{K^+ \times K^+} = -\int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k(c)\, \mathrm{d}c \cdot \mathbf{I}_{K^+ \times K^+},$$

where $\mathbf{I}_{K^+ \times K^+}$ is the $K^+ \times K^+$ identity matrix. Then we construct a $K \times K^- K^+$ matrix:

$$\mathbf{A}_{K \times K^- K^+} = \begin{bmatrix} \mathbf{B}^1_{K^- \times K^+} & \dots & \mathbf{B}^{K^-}_{K^- \times K^+} \\ \mathbf{C}^1_{K^+ \times K^+} & \dots & \mathbf{C}^{K^-}_{K^+ \times K^+} \end{bmatrix}.$$

Also, define the $K^- K^+$-dimentional and K-dimentional column vectors:

$$\boldsymbol{\lambda}_{K^- K^+ \times 1} = \begin{bmatrix} \lambda_{1,1} \\ \vdots \\ \lambda_{1,K^+} \\ \hline \vdots \\ \vdots \\ \hline \lambda_{k,1} \\ \vdots \\ \lambda_{k,K^+} \\ \hline \vdots \\ \vdots \\ \hline \lambda_{K^-,1} \\ \vdots \\ \lambda_{K^-,K^+} \end{bmatrix} \quad \text{and } \mathbf{b}_{K \times 1} = \begin{bmatrix} -f(1) \int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g'_1(c)\, \mathrm{d}c \\ \vdots \\ -f(k) \int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g'_k(c)\, \mathrm{d}c \\ \vdots \\ -f(K) \int_{s*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g'_K(c)\, \mathrm{d}c \end{bmatrix}.$$

Step 2. We want to establish that for all K-dimentional column vector $\mathbf{y}_{K\times 1}$ such that $(\mathbf{A}_{K\times K-K^+})^T\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K-K^+\times 1}$, it must be true that $(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} \geqslant 0$.

Inequality $(\mathbf{A}_{K\times K-K^+})^T\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K-K^+\times 1}$ is equivalent to

$$y_k \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k(c)\,\mathrm{d}c - y_{k'} \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k(c)\,\mathrm{d}c \geqslant 0,$$

namely, $y_k \geqslant y_{k'}$, for all $k \in \kappa^-$ and $k' \in \kappa^+$.

Define $\bar{y} \equiv \max\limits_{k'\in\kappa^+} y_{k'}$. Hence,

$$\begin{aligned}
&(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} \\
&= -\sum_{k\in\{1,\dots,K\}} y_k f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c \\
&= -\sum_{k\in\kappa^-} \underbrace{y_k}_{\geqslant \bar{y}} f(k) \underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c}_{\leqslant 0} - \sum_{k\in\kappa^+} \underbrace{y_k}_{\leqslant \bar{y}} f(k) \underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c}_{\geqslant 0} \\
&\geqslant -\sum_{k\in\kappa^-} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c - \sum_{k\in\kappa^+} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c \\
&= -\bar{y} \sum_{k\in\{1,\dots,K\}} f(k) \underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)g_k'(c)\,\mathrm{d}c}_{=0} = 0.
\end{aligned}$$

Step 3. Apply Farkas Lemma: either $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} = \mathbf{b}_{K\times 1}$ has a solution $\boldsymbol{\lambda}_{K-K^+\times 1} \geqslant \mathbf{0}_{K-K^+\times 1}$, or $(\mathbf{A}_{K\times K-K^+})^T\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K-K^+\times 1}$ has a solution $\mathbf{y}_{K\times 1}$ with $(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} < 0$. Given Step 2, we know that there exists a nonnegative vector $\boldsymbol{\lambda}_{K-K^+\times 1}$ such that $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} = \mathbf{b}_{K\times 1}$. Elements in the vector $\boldsymbol{\lambda}$ are the Lagrangian multipliers $(\lambda_{k,k'} \geqslant 0)_{k\in\kappa^-,k'\in\kappa^+}$.

Step 4. With the $(\lambda_{k,k'} \geqslant 0)_{k\in\kappa^-,k'\in\kappa^+}$ identified in Step 3, $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} = \mathbf{b}_{K\times 1}$ implies

$$\begin{aligned}
&\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)\left[f(k)g_k'(c) + \sum_{k'\in\kappa^+} \lambda_{k,k'} g_k(c)\right]\mathrm{d}c = 0, \forall k \in \kappa^-, \\
&\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*)\left[f(k')g_{k'}'(c) - \sum_{k\in\kappa^-} \lambda_{k,k'} g_k(c)\right]\mathrm{d}c = 0, \forall k' \in \kappa^+.
\end{aligned}$$

Given the above equations and the DSCP of the term in each square bracket established after setting up the Lagrangian, when we evaluate any of these integral expressions at some $s > s^*$ (resp. $s < s^*$) instead of s^*, the integral is nonpositive (resp. nonnegative). Hence, among all $s \in C$, $s = s^*$ not only satisfies the first order condition but also the second order condition, *i.e.*, $U_{(0,s^*)} \in \mathcal{U}$ maximizes $\mathcal{L}_k$ for each $k \in \{1, ..., K\}$ in the relaxed problem.

Case 2. Suppose $s^* = 1$.

We follow the construction of κ^- and κ^+ as in Case 1.

Recall the remark in Section 4.2, since $s^* = 1$, it must be the case that $\sum_{k \in \{1,...,K\}} f(k) g_k'(c) \geqslant 0, \forall c \in C$, which implies that $\sum_{k \in \{1,...,K\}} f(k) g_k'(1) \geqslant 0$. Similarly, for each $k \in \kappa^+$, it must be the case that $g_k'(1) \geqslant 0$. We can further show by contrapositive that for each $k \in \{1, ..., K\}$ such that $s_k^* < 1$, it must be the case that $g_k'(1) \leqslant 0$.

Step 1. We keep matrix $\boldsymbol{\lambda}_{K^- K^+ \times 1}$ and modify matrices $\mathbf{B}^k_{K^- \times K^+}$, $\mathbf{C}^k_{K^+ \times K^+}$, $\mathbf{A}_{K \times K^- K^+}$, and $\mathbf{b}_{K \times 1}$ used in Case 1 into the following:

$$\mathbf{B}^k_{K^- \times K^+} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ g_k(1) & \dots & g_k(1) \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow k\text{-th row} \\ \\ \\ \\ \end{matrix} \quad , \qquad \mathbf{C}^k_{K^+ \times K^+} = -g_k(1) \cdot \mathbf{I}_{K^+ \times K^+},$$

$$\mathbf{A}_{K \times K^- K^+} = \begin{bmatrix} \mathbf{B}^1_{K^- \times K^+} & \dots & \mathbf{B}^{K^-}_{K^- \times K^+} \\ \mathbf{C}^1_{K^+ \times K^+} & \dots & \mathbf{C}^{K^-}_{K^+ \times K^+} \end{bmatrix}, \qquad \mathbf{b}_{K \times 1} = \begin{bmatrix} -f(1) g_1'(1) \\ \vdots \\ -f(k) g_k'(1) \\ \vdots \\ -f(K) g_K'(1) \end{bmatrix}.$$

Step 2. We want to establish that for all K-dimentional column vector $\mathbf{y}_{K \times 1} \geqslant \mathbf{0}_{K \times 1}$ such that $(\mathbf{A}_{K \times K^- K^+})^T \mathbf{y}_{K \times 1} \leqslant \mathbf{0}_{K^- K^+ \times 1}$, it must be true that $(\mathbf{b}_{K \times 1})^T \mathbf{y}_{K \times 1} \leqslant 0$.

Inequality $(\mathbf{A}_{K\times K-K^+})^T\mathbf{y}_{K\times 1} \leqslant \mathbf{0}_{K-K^+\times 1}$ is equivalent to $y_k g_k(1) - y_{k'} g_k(1) \leqslant 0$. Hence, $0 \leqslant y_k \leqslant y_{k'}$, for all $k \in \kappa^-$ and $k' \in \kappa^+$. Define $\bar{y} \geqslant 0$ below:

$$\bar{y} = \begin{cases} y_1 \leqslant \min_{k\in\kappa^+} y_k, & \text{if } \kappa^- = \{1\}, \\ \min_{k\in\kappa^+} y_k, & \text{otherwise.} \end{cases}$$

If $\kappa^- = \{1\}$, it is obvious that $-\sum_{k\in\kappa^-} y_k f(k) g_k'(1) = -\sum_{k\in\kappa^-} \bar{y} f(k) g_k'(1)$. Otherwise, we have $s_k^* < s^* = 1$ for all $k \in \kappa^-$ and $-\sum_{k\in\kappa^-} \underbrace{y_k}_{\leqslant \bar{y}} f(k) \underbrace{g_k'(1)}_{\leqslant 0} \leqslant -\sum_{k\in\kappa^-} \bar{y} f(k) g_k'(1)$. These observations lead to the first inequality below:

$$\begin{aligned}
(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} &= -\sum_{k\in\{1,\ldots,K\}} y_k f(k) g_k'(1) \\
&= -\sum_{k\in\kappa^-} y_k f(k) g_k'(1) - \sum_{k\in\kappa^+} \underbrace{y_k}_{\geqslant \bar{y}} f(k) \underbrace{g_k'(1)}_{\geqslant 0} \\
&\leqslant -\sum_{k\in\kappa^-} \bar{y} f(k) g_k'(1) - \sum_{k\in\kappa^+} \bar{y} f(k) g_k'(1) \\
&= -\underbrace{\bar{y}}_{\geqslant 0} \underbrace{\sum_{k\in\{1,\ldots,K\}} f(k) g_k'(1)}_{\geqslant 0} \leqslant 0.
\end{aligned}$$

Step 3. Apply a variant of Farkas Lemma: either $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} \geqslant \mathbf{b}_{K\times 1}$ has a solution $\boldsymbol{\lambda}_{K-K^+\times 1} \geqslant \mathbf{0}_{K-K^+\times 1}$, or $(\mathbf{A}_{K\times K-K^+})^T\mathbf{y}_{K\times 1} \leqslant \mathbf{0}_{K-K^+\times 1}$ has a solution $\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K\times 1}$ with $(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} > 0$. Given Step 2, we now know that there exists a nonnegative vector $\boldsymbol{\lambda}_{K-K^+\times 1}$ such that $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} \geqslant \mathbf{b}_{K\times 1}$.

Step 4. By Step 3, there exists $(\lambda_{k,k'} \geqslant 0)_{k\in\kappa^-,k'\in\kappa^+}$, such that $\mathbf{A}_{K\times K-K^+}\boldsymbol{\lambda}_{K-K^+\times 1} \geqslant \mathbf{b}_{K\times 1}$, *i.e.*,

$$\begin{aligned}
& f(k) g_k'(1) + \sum_{k'\in\kappa^+} \lambda_{k,k'} g_k(1) \geqslant 0, \forall k \in \kappa^-, \\
& f(k') g_{k'}'(1) - \sum_{k\in\kappa^-} \lambda_{k,k'} g_k(1) \geqslant 0, \forall k' \in \kappa^+.
\end{aligned}$$

Because of the DSCP established after setting up the Lagrangian, we further know that

$$f(k)g_k'(c) + \sum_{k' \in \kappa^+} \lambda_{k,k'} g_k(c) \geqslant 0, \forall k \in \kappa^-, c \in C,$$
$$f(k')g_{k'}'(c) - \sum_{k \in \kappa^-} \lambda_{k,k'} g_k(c) \geqslant 0, \forall k' \in \kappa^+, c \in C,$$

As a result, $U_{(0,s^*=1)}$ maximizes $\mathcal{L}_k$ for all $k \in \{1, ..., K\}$.

Case 3. Suppose $s^* = 0$.

In this case,

$$\sum_{k \in \{1,\dots,K\}} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k'(c) \,\mathrm{d}c \leqslant 0.$$

We now modify the construction of sets κ^- and κ^+: let $\kappa^- \equiv \{1\} \cup \{k \in \{2, ...K-1\} : s_k^* = 0\}$ and $\kappa^+ \equiv \{K\} \cup \{k \in \{2, ...K-1\} : s_k^* > 0\}$.

We follow Step 1 in Case 1 and only adjust Steps 2 to 4.

Step 2. We want to establish that for all K-dimentional column vector $\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K\times 1}$ such that $(\mathbf{A}_{K\times K^- K^+})^T \mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K^- K^+ \times 1}$, it must be true that $(\mathbf{b}_{K\times 1})^T \mathbf{y}_{K\times 1} \geqslant 0$.

Fix any K-dimentional column vector $\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K\times 1}$ such that $(\mathbf{A}_{K\times K^- K^+})^T \mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K^- K^+ \times 1}$. Inequality $(\mathbf{A}_{K\times K^- K^+})^T \mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K^- K^+ \times 1}$ is equivalent to

$$y_k \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k(c) \,\mathrm{d}c - y_{k'} \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k(c) \,\mathrm{d}c \geqslant 0,$$

namely, $y_k \geqslant y_{k'}$, for all $k \in \kappa^-$ and $k' \in \kappa^+$.

Define $\bar{y} \geqslant 0$ below:

$$\bar{y} = \begin{cases} y_K \leqslant \min_{k \in \kappa^-} y_k, & \text{if } \kappa^+ = \{K\}, \\ \min_{k \in \kappa^-} y_k, & \text{otherwise.} \end{cases}$$

If $\kappa^+ = \{K\}$, it is obvious that

$$-\sum_{k \in \kappa^+} y_k f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k'(c) \,\mathrm{d}c = -\sum_{k \in \kappa^+} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k'(c) \,\mathrm{d}c.$$

Otherwise, $s_k^* > s^* = 0$ for all $k \in \kappa^+$, which further implies that

$$-\sum_{k \in \kappa^+} \underbrace{y_k}_{\leqslant \bar{y}} f(k) \underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k'(c) \,\mathrm{d}c}_{\geqslant 0} \geqslant -\sum_{k \in \kappa^+} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega > s^*]} (c - s^*) g_k'(c) \,\mathrm{d}c.$$

These observations lead to the first inequality below:

$$
\begin{aligned}
&(\mathbf{b}_{K\times 1})^T \mathbf{y}_{K\times 1} \\
&= -\sum_{k\in\kappa^-} \underbrace{y_k}_{\geqslant \bar{y}} f(k) \underbrace{\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) g_k'(c)\,\mathrm{d}c}_{\leqslant 0} - \sum_{k\in\kappa^+} y_k f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) g_k'(c)\,\mathrm{d}c \\
&\geqslant -\sum_{k\in\kappa^-} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) g_k'(c)\,\mathrm{d}c - \sum_{k\in\kappa^+} \bar{y} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) g_k'(c)\,\mathrm{d}c \\
&= -\underbrace{\bar{y}}_{\geqslant 0} \underbrace{\sum_{k\in\{1,\ldots,K\}} f(k) \int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) g_k'(c)\,\mathrm{d}c}_{\leqslant 0} \geqslant 0.
\end{aligned}
$$

Step 3. Apply a variant of Farkas Lemma: either $\mathbf{A}_{K\times K^-K^+}\boldsymbol{\lambda}_{K^-K^+\times 1} \leqslant \mathbf{b}_{K\times 1}$ has a solution $\boldsymbol{\lambda}_{K^-K^+\times 1} \geqslant \mathbf{0}_{K^-K^+\times 1}$, or $(\mathbf{A}_{K\times K^-K^+})^T\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K^-K^+\times 1}$ has a solution $\mathbf{y}_{K\times 1} \geqslant \mathbf{0}_{K\times 1}$ with $(\mathbf{b}_{K\times 1})^T\mathbf{y}_{K\times 1} < 0$. Given Step 2, we now know that there exists a nonnegative vector $\boldsymbol{\lambda}_{K^-K^+\times 1}$ such that $\mathbf{A}_{K\times K^-K^+}\boldsymbol{\lambda}_{K^-K^+\times 1} \leqslant \mathbf{b}_{K\times 1}$.

Step 4. With the $(\lambda_{k,k'} \geqslant 0)_{k\in\kappa^-,k'\in\kappa^+}$ identified in Step 3, $\mathbf{A}_{K\times K^-K^+}\boldsymbol{\lambda}_{K^-K^+\times 1} \leqslant \mathbf{b}_{K\times 1}$ implies

$$
\begin{aligned}
&\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) \left[f(k)g_k'(c) + \sum_{k'\in\kappa^+} \lambda_{k,k'} g_k(c) \right] \mathrm{d}c \leqslant 0, \forall k \in \kappa^-, \\
&\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]} (c-s^*) \left[f(k')g_{k'}'(c) - \sum_{k\in\kappa^-} \lambda_{k,k'} g_k(c) \right] \mathrm{d}c \leqslant 0, \forall k' \in \kappa^+.
\end{aligned}
$$

The above inequalities and the DSCP of each term in the square bracket imply that $U_{(0,s^*=0)} \in \mathcal{U}$ maximizes $\mathcal{L}_k$ for each $k \in \{1,\ldots,K\}$ in the relaxed problem. □

B.2 Sufficient condition for discriminatory disclosure

The following proposition provides a sufficient condition for non-discriminatory disclosure. The negation of this condition serves as a sufficient condition for discriminatory disclosure.

Proposition 3. *For each $k \in \{2,\ldots,K\}$, let*

$$\bar{g}_k(c) \equiv \frac{f(k)g_k(c) + \ldots + f(K)g_K(c)}{f(k) + \ldots + f(K)}.$$

The optimal persuasion mechanism can be implemented by an experiment only if for each $k \in \{2, ..., K\}$, $c_1 < s^*$, *and* $c_2 > s^*$,

$$\frac{\bar{g}_k'(c_1)}{g_{k-1}(c_1)} \geqslant -\frac{f(1)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_1'(c)\,\mathrm{d}c + ... + f(k-1)\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_{k-1}'(c)\,\mathrm{d}c}{(f(k)+...+f(K))\int_{s^*}^{\mathbb{E}[\omega|\omega>s^*]}(c-s^*)g_{k-1}(c)\,\mathrm{d}c}$$
$$\geqslant \frac{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)\bar{g}_k'(c)\,\mathrm{d}c}{\int_{c_2}^{\mathbb{E}[\omega|\omega>s^*]}(c-c_2)g_{k-1}(c)\,\mathrm{d}c}. \tag{20}$$

Proof of Proposition 3. Supposes there exists $k \in \{2, ..., K\}$, $c_1 < s^*$, or $c_2 > s^*$ such that (20) is violated. One follow the proof of Proposition 1 and design a menu of experiments such that t_1 to t_{k-1} receive the new experiment designed for the L type therein and t_k to t_K receive the new experiment designed for the H type such that type $k-1$ is indifferent between the two new experiments.

By Assumption 2 (stated for $K \geqslant 3$), Lemma 4, and the indifference condition for type $k-1$, every type k' with $k' \neq k$ will weakly prefer the experiment designed for him. As a result, IC_1 holds for the newly constructed menu.

By following the argument of the two-type case, one can easily show that the above menu strictly benefits the principal. □