Online Appendix to "Collusion-proof Mechanisms for Full Surplus Extraction"

Analysis Under an Alternative Collusion-proofness Notion

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In this Online Appendix, we discuss the alternative collusion-proofness notion motivated by Safronov (2018). An ambiguous mechanism (q, T) is said to satisfy the **coalition incentive compatibility** (CIC) condition, if for all $S \in 2^I \backslash \{\varnothing\}$ with $|S| \geqslant 2$, $\theta_S \in \Theta_S$, and $\delta^S : \Theta_S \to \Delta(\Theta_S)$, $V_S[q,T](\theta_S, \bar{\delta}^S) \geqslant V_S[q,T](\theta_S, \delta^S)$, where $V_S[q,T](\theta_S, \delta^S) = \min_{t \in T} V_S[q,t](\theta_S, \delta^S)$. Namely, we view each coalition S as a pseudo agent whose type is an element in Θ_S and whose utility is the sum of its members' utility levels. Each pseudo agent also uses the MEU to compute his interim payoff. The CIC condition requires that no pseudo agent has the incentive to misreport.

The CIC condition differs from the RCP (or RCP*) condition in the main text in a few ways. First, agents in a coalition are assumed to pool their private information under the CIC condition. However, the mediator of coalition S in the RCP notion has to rely on an S-side contract to elicit members' private information. Second, a coalition is implicitly assumed to engage in manipulation under the CIC condition if there is a joint reporting strategy δ^S and a type profile θ_S such that θ_S strictly benefits from adopting δ^S. On the other hand, the RCP notion requires the S-side contract to be S-IR, i.e., to be weakly profitable for every $i \in S$ and $\theta_i \in \Theta_i$. Moreover, the minimization operators are imposed at different stages under the two notions: it is imposed on the expected payoff of type-θ_S pseudo agent S in the CIC notion, but imposed on the expected payoff of type-θ_i agent $i \in S$ in the RCP condition. These differences make neither condition stronger than the other. Hence, the design of a

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CIC ambiguous mechanism remains a non-trivial question. Despite the differences, we study both the RCP condition (in the main text) and the CIC condition since the two notions are representative ways to study collusion-proofness in the literature.[16]

Lemma 1 directly implies Proposition 3, which can be viewed as the counterpart of Proposition 1 under the CIC condition. We thus omit its proof. Note that the following result does not depend on the cardinality of Θ.

Proposition 3. *No information structure (Θ, p) can guarantee FSE via standard Bayesian mechanisms satisfying the CIC condition.*

We establish the counterpart of Proposition 2 under the CIC condition.

Proposition 4. *Given an information structure (Θ, p), the following statements are equivalent:*

1. *The CBDP property holds for prior p.*
2. *The information structure (Θ, p) guarantees FSE via ambiguous mechanisms satisfying the CIC condition.*

We remark on how to establish Statement 2 from Statement 1 before presenting the proof. To do this, we establish the following lemma.

Lemma 7. *Given $S \in 2^I \backslash \{\varnothing, I\}$ and type profile $\bar{\theta}_S \in \Theta_S$, if there does not exist $\hat{\theta}_S \in \Theta_S \backslash \{\bar{\theta}_S\}$ such that $p(\cdot|\hat{\theta}_S) = p(\cdot|\bar{\theta}_S)$, then there exists a transfer rule $\phi^{\bar{\theta}_S} : \Theta \to \mathbb{R}^n$ such that*

(i) $\sum_{i \in C} \sum_{\theta_{-C} \in \Theta_{-C}} \phi_i^{\bar{\theta}_S}(\theta_C, \theta_{-C}) p(\theta_{-C}|\theta_C) = 0$ *for all $C \in 2^I \backslash \{\varnothing\}$ and $\theta_C \in \Theta_C$;*

[16] One can think of other notions that possess similarities with both the RCP condition and the CIC condition. For example, suppose members in S are forced to accept any S-IC ambiguous S-side contract (δ^S, Ψ^S) and the mediator can choose to misreport to the main mechanism on behalf of S when it is profitable for every member. Then a collusion-proofness notion could require that there does not exist a coalition S, an S-IC ambiguous S-side contract (δ^S, Ψ^S), and $\theta_S \in \Theta_S$ such that

$$\min_{t \in T, \psi^S \in \Psi^S} \sum_{\theta'_S \in \Theta_S} \sum_{\theta_{-S} \in \Theta_{-S}} [u_i(q(\theta'_S, \theta_{-S}), (\theta_S, \theta_{-S})) + t_i(\theta'_S, \theta_{-S}) + \psi_i^S(\theta_S)] p(\theta_{-S}|\theta_S) \delta^S[\theta_S](\theta'_S)$$
$$> \min_{t \in T} \sum_{\theta_{-S} \in \Theta_{-S}} [u_i(q(\theta_S, \theta_{-S}), (\theta_S, \theta_{-S})) + t_i(\theta_S, \theta_{-S})] p(\theta_{-S}|\theta_S), \forall i \in S.$$

(ii) $\sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i^{\bar{\theta}_S}(\hat{\theta}_S, \theta_{-S}) p(\theta_{-S}|\bar{\theta}_S) < 0$ *for all* $\hat{\theta}_S \in \Theta_S \backslash \{\bar{\theta}_S\}$.

We sketch the proof of the lemma first. As coalitions and individuals overlap with each other, (i) above imposes multiple intertwined constraints on the transfer rule $\phi^{\bar{\theta}_S}$ and complicates the problem compared to the case without coalition concerns. As a simplification, we first view the problem as a two-agent one, which contains pseudo agents S and $I \backslash S$ only. In Step 1 of the proof, we apply the transposition theorem of Motzkin (1951) to establish the existence of an ex-post budget balanced transfer rule ϕ between the two pseudo agents for which (ii) and (i) with respect to $C = S$ and $C = I \backslash S$ hold. In Steps 2 and 3, we redistribute $(\phi_i)_{i \in S}$ among agents in S, redistribute $(\phi_i)_{i \in I \backslash S}$ among agents in $I \backslash S$, so that under the redistribution $\hat{\phi}$, the equality in (i) holds for all other C. The division has to be carefully designed rather than a simplistic equal division. In particular, when specifying $\hat{\phi}_i$, not only the equation in (i) with respect to $C = \{i\}$ is affected, those with respect to any other coalition containing i are also affected. To tackle this challenge, we apply the alternative theorem of Fredholm (1903) to establish the existence of a division such that every agent $i \in I$ receives zero in expectation conditional on any θ_{-i}. Intuitively, this means that the pseudo agent $I \backslash \{i\}$ always believes that i receives zero in expectation. In Steps 4 to 6, we show that the division satisfies all required conditions by applying the law of iterated expectations.

An ambiguous mechanism (q, T), where $T \equiv \{\eta + \lambda \phi^{\bar{\theta}_S} | S \in 2^I \backslash \{\emptyset, I\}, \bar{\theta}_S \in \Theta_S\}$, η is the same with the one constructed in Step 1 of Proposition 2, and $\lambda \in \mathbb{R}_+$ is sufficiently large, is then shown to be feasible, extract the full surplus, and satisfy the CIC condition. To see this, by (i) in Lemma 7, each $\phi^{\bar{\theta}_S}$ does not affect the MD's ex-post payoff, as setting $C = I$ implies that $\phi^{\bar{\theta}_S}$ is ex-post budget balanced; neither does $\phi^{\bar{\theta}_S}$ affect any (potentially pseudo) agent's on-path interim payoff, as $\phi^{\bar{\theta}_S}$ gives every $C \in 2^I \backslash \{\emptyset, I\}$ zero expected utility on path. However, by (ii) above, for each type-θ_S (potentially pseudo) agent $S \in 2^I \backslash \{\emptyset, I\}$, any unilateral deviation from truthful report (in potentially mixed strategy) earns him a negative expected transfer under ϕ^{θ_S}. When the multiplier λ is sufficiently large, $\eta + \lambda \phi^{\theta_S} \in T$ gives θ_S a negative expected utility, which bounds his MEU of misreporting from above and eventually establishes the IC and CIC conditions.[17]

[17] Such a construction also works for the collusion-proofness notion mentioned in footnote 16.

Before presenting the details, we remark that although the above construction of (q, T) satisfies CIC, it may not satisfy RCP/RCP*. Intuitively, the above construction ensures that by misreporting, a non-grand pseudo agent S earns a low MEU after knowing θ_S. However, S may find it profitable to misreport without conditional on any θ_S, which leaves room for an S-feasible reallocational manipulation. The inconsistency happens because randomization may be used to partially hedge against ambiguity, and the ex-ante utility (the one without conditional on θ_S) of misreporting as randomization of interim utilities (those after knowing θ_S) is not necessarily low.

However, we can let $\hat{T} \equiv T^1 \cup T^2$, where $T^1 \equiv \{\eta + \lambda_1 \phi^{\tilde{\theta}_i} | i \in I, \tilde{\theta}_i \in \Theta_i\} \cup \{\eta + \lambda_2 \phi^S | S \in 2^I \backslash \{\varnothing, I\} \text{ with } 2 \leqslant |S| \leqslant n-1\}$ and $T^2 = \{\eta + \lambda_3 \phi^{\theta_S} | S \in 2^I \backslash \{\varnothing, I\} \text{ with } 2 \leqslant |S| \leqslant n-1, \theta_S \in \Theta_S\}$. Let the choice of λ_1 and λ_2 follow from the proof of Proposition 2, and let $\lambda_3 \in \mathbb{R}_+$ be weakly larger than

$$\max_{\substack{S \in 2^I \backslash \{\varnothing, I\}, \\ \bar{\theta}_S, \hat{\theta}_S \in \Theta_S \text{ with } \bar{\theta}_S \neq \hat{\theta}_S}} \frac{\min\{V_S[q, T^1](\bar{\theta}_S, \bar{\theta}_S), V_S[q, \eta](\theta_S, \theta_S)\} - V_S[q, \eta](\bar{\theta}_S, \hat{\theta}_S)}{\sum\limits_{i \in S} \sum\limits_{\theta_{-S} \in \Theta_{-S}} \phi_i^{\bar{\theta}_S}(\hat{\theta}_S, \theta_{-S}) p(\theta_{-S} | \bar{\theta}_S)}.$$

Then one can follow the proofs of Propositions 2 and 4 and show that $(q, \hat{T})$ is an FSE ambiguous mechanism that simultaneously satisfies the RCP* condition and the CIC condition.

We now include the details of the proofs of Lemma 7 and Proposition 4.

Proof of Lemma 7. **Step 1.** Show that there exists an ex-post budget balanced transfer rule $\phi : \Theta \to \mathbb{R}^n$ such that

(a) $\sum_{i \in C} \sum_{\theta_{-C} \in \Theta_{-C}} \phi_i(\theta_C, \theta_{-C}) p(\theta_{-C} | \theta_C) = 0$ for all $C \in \{S, I \backslash S\}$ and $\theta_C \in \Theta_C$;

(b) $\sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i(\hat{\theta}_S, \theta_{-S}) p(\theta_{-S} | \bar{\theta}_S) < 0$ for all $\hat{\theta}_S \in \Theta_S \backslash \{\bar{\theta}_S\}$.

To see this, recall the version of Lemma 4 where the coalition in the statement of that lemma is equal to the set of all agents. By viewing the current environment as one with two "pseudo" agents, one "pseudo" agent S, and the other "pseudo" agent $I \backslash S$, we can follow Lemma 4 to establish the above result.

Step 2. Given the above ϕ, prove the existence of $(\hat{\phi}_i : \Theta \to \mathbb{R})_{i \in S}$ such that

(a) $\sum_{\theta_i \in \Theta_i} \hat{\phi}_i(\theta_i, \theta_{-i}) p(\theta_i | \theta_{-i}) = 0$ for all $i \in S$ and $\theta_{-i} \in \Theta_{-i}$;

(b) $\sum_{i\in S}\hat{\phi}_i(\theta) = \sum_{i\in S}\phi_i(\theta)$ for all $\theta\in\Theta$.

When $|S| = 1$, denote the agent in S by i. Let $\hat{\phi}_i = \phi_i$. Then (b) is satisfied and

$$\sum_{\theta_i\in\Theta_i}\hat{\phi}_i(\theta_i,\theta_{-i})p(\theta_i|\theta_{-i}) = \sum_{\theta_i\in\Theta_i}\phi_i(\theta_i,\theta_{-i})p(\theta_i|\theta_{-i})$$
$$\overset{\text{ex-post budget balanced}\,\phi}{=} -\sum_{\theta_i\in\Theta_i}\sum_{j\in I\setminus\{i\}}\phi_j(\theta_i,\theta_{-i})p(\theta_i|\theta_{-i}) \overset{\text{Step 1}}{=} 0$$

for each $\theta_{-i}\in\Theta_i$, i.e., (a) is satisfied.

Now we proceed with the case that $|S|\geqslant 2$. Suppose by way of contradiction that there does not exist $\hat{\phi}$ such that (a) and (b) are satisfied.

To apply Theorem 2, with the vectors defined in Section A.1, we construct matrices B and b of dimensions $m\times l$ and $m\times 1$, respectively, where $m = \sum_{i\in S}|\Theta_{-i}| + |\Theta|$ and $l = n|\Theta|$. Matrix B is obtained by vertically stacking up $\sum_{i\in S}|\Theta_{-i}|$ row vectors $p^{\{i\}}_{\theta_{-i}\theta_{-i}}\in\mathbb{R}^l_+$ for all $i\in S$ and $\theta_{-i}\in\Theta_{-i}$ (the order does not matter), and $|\Theta|$ row vectors e^S_θ for all $\theta\in\Theta$ (following the order of elements in Θ). Construct matrix b by vertically stacking up $\sum_{i\in S}|\Theta_{-i}|$ zeros and $|\Theta|$ numbers $\sum_{i\in S}\phi_i(\theta)$ for all $\theta\in\Theta$ (following the order of elements in Θ). The supposition above implies that $Bx = b$ has no column vector solution $x\in\mathbb{R}^l$.

By Theorem 2, $B'y = \mathbf{0}_{l\times 1}$ has a column vector solution $y\in\mathbb{R}^m$ with $y'b\neq 0$, i.e., there exists a profile of numbers $(a_{\theta_{-i}}\in\mathbb{R})_{\theta_{-i}\in\Theta_{-i},i\in S}$ and a profile $(b_\theta\in\mathbb{R})_{\theta\in\Theta}$ such that

$$\sum_{i\in S}\sum_{\theta_{-i}\in\Theta_{-i}} a_{\theta_{-i}}p^{\{i\}}_{\theta_{-i}\theta_{-i}} + \sum_{\theta\in\Theta} b_\theta e^S_\theta = \mathbf{0}_{1\times l}, \tag{26}$$

$$\sum_{i\in S}\sum_{\theta\in\Theta} b_\theta\phi_i(\theta)\neq 0. \tag{27}$$

Recall from Appendix A.1, each side of (26) is a vector in $\mathbb{R}^l$ and each dimension corresponds to an agent and a type profile.

We now establish that

$$a_{\theta_{-i}} = a_{\theta_{-j}}, \forall\theta\in\Theta, i,j\in S\,\text{with}\, i\neq j. \tag{28}$$

To see this, fix any $\theta\in\Theta$ and $i,j\in S$ with $i\neq j$ for now. The dimensions in (26) corresponding to i and θ imply that

$$a_{\theta_{-i}}p(\theta) + b_\theta = 0. \tag{29}$$

Similarly, the dimensions in (26) corresponding to j and θ imply that

$$a_{\theta_{-j}}p(\theta) + b_\theta = 0.$$

Given $p(\theta) > 0$, the above two expressions imply that $a_{\theta_{-i}} = a_{\theta_{-j}}$.

We now take Steps (i) to (iii) to show that for any $\theta_{-S} \in \Theta_{-S}$, there exists a unique constant $\kappa_{\theta_{-S}}$ such that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = \kappa_{\theta_{-S}}$ for all $\theta_S \in \Theta_S$ and $i \in S$.

Step (i). We begin with $\theta_S \neq \theta'_S$ that differ from each other at exactly one agent in S only and show that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{i\}},\theta_{-S})}$ for all $i \in S$. To this see, label this agent for whom θ_S and θ'_S differ by j. As θ_S and θ'_S differ only at j, $(\theta_{S\setminus\{j\}}, \theta_{-S}) = (\theta'_{S\setminus\{j\}}, \theta_{-S})$, and thus, $a_{(\theta_{S\setminus\{j\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{j\}},\theta_{-S})}$. This further implies that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} \overset{(28)}{=} a_{(\theta_{S\setminus\{j\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{j\}},\theta_{-S})} \overset{(28)}{=} a_{(\theta'_{S\setminus\{i\}},\theta_{-S})}$ for all $i \in S$ with $i \neq j$. To this end, we have shown that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{i\}},\theta_{-S})}$ for all $i \in S$ (including the special case that $i = j$).

Step (ii). We show that for any $\theta_S \neq \theta'_S$, $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{i\}},\theta_{-S})}$ for all $i \in S$. To see this, there exists a finite sequence $(\theta_S^m)_{m=1,\ldots,\bar{m}}$ where $\theta_S^1 = \theta_S$, $\theta_S^{\bar{m}} = \theta'_S$, and every pair of adjacent type profiles differ at exactly one agent. By applying the argument in Step (i) to each pair of adjacent type profiles recursively, it must be true that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = a_{(\theta'_{S\setminus\{i\}},\theta_{-S})}$ for all $i \in S$.

Step (iii). Finally, notice that $a_{(\theta_{S\setminus\{j\}},\theta_{-S})} \overset{(28)}{=} a_{(\theta_{S\setminus\{i\}},\theta_{-S})} \overset{\text{Step }(ii)}{=} a_{(\theta'_{S\setminus\{i\}},\theta_{-S})} \overset{(28)}{=} a_{(\theta'_{S\setminus\{j\}},\theta_{-S})}$ for any θ_S, θ'_S and $i, j \in S$ with $i \neq j$. Hence, there exists a unique constant $\kappa_{\theta_{-S}}$ such that $a_{(\theta_{S\setminus\{i\}},\theta_{-S})} = \kappa_{\theta_{-S}}$ for all $\theta_S \in \Theta_S$ and $i \in S$.

The above observation, joint with (29), implies that for each $\theta_{-S} \in \Theta_{-S}$, $\kappa_{\theta_{-S}}p(\theta_S, \theta_{-S}) + b_{(\theta_S,\theta_{-S})} = 0$, i.e., $b_{(\theta_S,\theta_{-S})} = -\kappa_{\theta_{-S}}p(\theta_S, \theta_{-S})$, for all $\theta_S \in \Theta_S$.

Notice that ϕ satisfies the requirements established in Step 1. We have

$$\sum_{i\in S}\sum_{\theta_S\in\Theta_S} p(\theta_S|\theta_{-S})\phi_i(\theta_S, \theta_{-S}) \overset{\text{ex-post budget balanced }\phi}{=} -\sum_{j\in I\setminus S}\sum_{\theta_S\in\Theta_S} p(\theta_S|\theta_{-S})\phi_j(\theta_S, \theta_{-S}) \overset{\text{(a) in Step 1}}{=} 0$$

for all $\theta_{-S} \in \Theta_{-S}$.

As a result,

$$\begin{aligned}
\sum_{i\in S}\sum_{\theta\in\Theta} b_\theta\phi_i(\theta) &= \sum_{\theta_{-S}\in\Theta_{-S}}\sum_{i\in S}\sum_{\theta_S\in\Theta_S} b_{(\theta_S,\theta_{-S})}\phi_i(\theta_S, \theta_{-S}) \\
&= \sum_{\theta_{-S}\in\Theta_{-S}} -\kappa_{\theta_{-S}}p(\theta_{-S}) \underbrace{\sum_{i\in S}\sum_{\theta_S\in\Theta_S} p(\theta_S|\theta_{-S})\phi_i(\theta_S, \theta_{-S})}_{=0} = 0,
\end{aligned}$$

a contradiction with expression (27).

Step 3. Similar to Step 2, given the ϕ in Step 1, one can establish the existence $(\hat{\phi}_i : \Theta \to \mathbb{R})_{i\in I\backslash S}$ such that

(a) $\sum_{\theta_i\in\Theta_i} \hat{\phi}_i(\theta_i, \theta_{-i})p(\theta_i|\theta_{-i}) = 0$ for all $i \in I\backslash S$ and $\theta_{-i} \in \Theta_{-i}$;

(b) $\sum_{i\in I\backslash S} \hat{\phi}_i(\theta) = \sum_{i\in I\backslash S} \phi_i(\theta)$ for all $\theta \in \Theta$.

Step 4. Combine $(\hat{\phi}_i : \Theta \to \mathbb{R})_{i\in S}$ and $(\hat{\phi}_i : \Theta \to \mathbb{R})_{i\in I\backslash S}$ into a transfer rule $\hat{\phi}$. By budget balance of ϕ and (b) from Steps 2 and 3, $\hat{\phi}$ also satisfies ex-post budget balance. As a result, we have established the existence of an ex-post budget balanced transfer rule $\hat{\phi} : \Theta \to \mathbb{R}^n$ such that,

(a) $\sum_{\theta_i\in\Theta_i} \hat{\phi}_i(\theta_i, \theta_{-i})p(\theta_i|\theta_{-i}) = 0$ for all $i \in I$ and $\theta_{-i} \in \Theta_{-i}$;

(b) $\sum_{i\in C} \hat{\phi}_i(\theta) = \sum_{i\in C} \phi_i(\theta)$ for all $C \in \{S, I\backslash S\}$ and $\theta \in \Theta$.

Step 5. Show that

$$\sum_{i\in C} \sum_{\theta_{-C}\in\Theta_{-C}} \hat{\phi}_i(\theta_C, \theta_{-C})p(\theta_{-C}|\theta_C) = 0, \forall C \in 2^I\backslash\{\varnothing, I\}, \theta_C \in \Theta_C. \tag{30}$$

By (a) and budget balance of $\hat{\phi}$ derived from Step 4,

$$\sum_{j\in I\backslash\{i\}} \sum_{\theta_i\in\Theta_i} \hat{\phi}_j(\theta_i, \theta_{-i})p(\theta_i|\theta_{-i}) = 0, \forall i \in I, \theta_{-i} \in \Theta_{-i}. \tag{31}$$

Hence, for any $C \in 2^I\backslash\{\varnothing, I\}$ with $|C| = n - 1$, the equation in (30) holds. It remains to show that the equation in (30) holds for any $C \in 2^I\backslash\{\varnothing, I\}$ with $|C| < n - 1$. Fix such a C and $K \in 2^I\backslash\{\varnothing, I\}$ such that $C \cap K = \varnothing$ and $|C \cup K| = n - 1$. By (31),

$$\sum_{i\in C\cup K} \sum_{\theta_{-C\cup K}\in\Theta_{-C\cup K}} \hat{\phi}_i(\theta_{C\cup K}, \theta_{-C\cup K})p(\theta_{-C\cup K}|\theta_{C\cup K}) = 0, \forall\theta_{C\cup K} \in \Theta_{C\cup K}.$$

Since $C \subseteq C \cup K$, the law of iterated expectations implies that

$$\sum_{i\in C\cup K} \sum_{\theta_{-C}\in\Theta_{-C}} \hat{\phi}_i(\theta_C, \theta_{-C})p(\theta_{-C}|\theta_C) = 0, \forall\theta_C \in \Theta_C. \tag{32}$$

For any $i \in K$, since $C \subseteq I \backslash \{i\}$, the law of iterated expectations and (a) from Step 4 imply

$$\sum_{\theta_{-C} \in \Theta_{-C}} \hat{\phi}_i(\theta_C, \theta_{-C}) p(\theta_{-C}|\theta_C) = 0, \; \forall \theta_C \in \Theta_C. \tag{33}$$

From (32) and the fact that (33) holds for all $i \in K$, we know that (30) holds.

Step 6. Rename $\hat{\phi}$ as $\phi^{\bar{\theta}_S}$, which satisfies the conditions required by the lemma. □

Proof of Proposition 4. To establish Statement 1 from Statement 2, we assume by way of contradiction that Statement 2 holds but the CBDP property fails. Then under the same payoff structure in Lemma 6 with $\epsilon \in (0, \frac{1}{n-1})$, we fix a coalition S and type profiles $\bar{\theta}_S \neq \hat{\theta}_S$ such that $i \in S$, $\bar{\theta}_i$ is a component of $\bar{\theta}_S$, $\hat{\theta}_i$ is a component of $\hat{\theta}_S$, and $\bar{\theta}_i \neq \hat{\theta}_i$. It is easy to show that summing up constraints $\text{CIC}(\bar{\theta}_S; \hat{\theta}_S)$ and $\text{CIC}(\hat{\theta}_S; \bar{\theta}_S)$ yields a contradiction. We omit the details. Now we take two steps to establish Statement 2 from Statement 1.

Step 1. Fix any efficient allocation rule $q : \Theta \to A$, and define $T = \{\eta + \lambda \phi^{\theta_S} | S \in 2^I \backslash \{\varnothing, I\}, \theta_S \in \Theta_S\}$, where η is constructed in the same way as in Proposition 2, each ϕ^{θ_S} satisfies the conditions in Lemma 7, and $\lambda \in \mathbb{R}_+$ is weakly larger than

$$\max_{\substack{S \in 2^I \backslash \{\varnothing, I\}, \\ \bar{\theta}_S, \hat{\theta}_S \in \Theta_S \text{ with } \bar{\theta}_S \neq \hat{\theta}_S}} \frac{V_S[q,\eta](\bar{\theta}_S, \bar{\theta}_S) - V_S[q,\eta](\bar{\theta}_S, \hat{\theta}_S)}{\sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i^{\bar{\theta}_S}(\hat{\theta}_S, \theta_{-S}) p(\theta_{-S}|\bar{\theta}_S)}.$$

Step 2. It is easy to verify that (q, T) satisfies the IR condition and (4). To demonstrate CIC and IC, we discuss two cases.

Case 1, $S = I$. Since each $\phi^{\tilde{\theta}_C}$ is budget balanced, for each $t \in T$, $\theta \in \Theta$, and δ^I,

$$\begin{aligned} V_I[q,t](\theta, \delta^I) &= \sum_{\theta' \in \Theta} \sum_{i \in I} [u_i\big(q(\theta'), \theta\big) + \eta_i(\theta')] \delta^I[\theta](\theta') \\ &\overset{(21)}{=} \sum_{\theta' \in \Theta} [\sum_{i \in I} u_i\big(q(\theta'), \theta\big) + u_0\big(q(\theta')\big)] \delta^I[\theta](\theta') - FS \\ &\leqslant \sum_{i \in I} u_i\big(q(\theta), \theta\big) + u_0\big(q(\theta)\big) - FS = V_I[q,t](\theta, \bar{\delta}^I), \end{aligned} \tag{34}$$

where the inequality follows from the efficiency of q. As a result, $V_I[q,T](\theta, \bar{\delta}^I) \geqslant V_I[q,T](\theta, \delta^I)$.

Case 2, $S \in 2^I \backslash \{\varnothing, I\}$. For each $\theta_S \in \Theta_S$ and $t \in T$, since $t = \eta + \lambda \phi^{\tilde{\theta}_C}$ for some $C \in 2^I \backslash \{\varnothing, I\}$ and $\tilde{\theta}_C \in \Theta_C$, Condition (i) of $\phi^{\tilde{\theta}_C}$ in Lemma 7 implies that

$$V_S[q,t](\theta_S, \bar{\delta}^S) = V_S[q,\eta](\theta_S, \bar{\delta}^S) = V_S[q,T](\theta_S, \bar{\delta}^S).$$

For each $\theta_S \in \Theta_S$, as $\eta + \lambda\phi^{\theta_S} \in T$, $V_S[q,T](\theta_S, \hat{\theta}_S)$ is no higher than

$$V_S[q,\eta](\theta_S, \hat{\theta}_S) + \lambda \sum_{i \in S} \sum_{\theta_{-S} \in \Theta_{-S}} \phi_i^{\theta_S}(\hat{\theta}_S, \theta_{-S}) p(\theta_{-S}|\theta_S) \leqslant V_S[q,\eta](\theta_S, \bar{\delta}^S), \forall \hat{\theta}_S \in \Theta_S,$$

where the inequality follows from the choice of λ and Condition (ii) of ϕ^{θ_S} stated in Lemma 7. Hence, $V_S[q,T](\theta_S, \delta^S) \leqslant V_S[q, \eta + \lambda\phi^{\theta_S}](\theta_S, \delta^S) \leqslant V_S[q,T](\theta_S, \bar{\delta}^S)$ for any δ^S.

To this end, we have completed Step 2. □

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